Math 116 — First Exam

October 10, 2007

Name: Exam Solutions

Instructor: ______________________ Section: ________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. There is a partial table of integrals, and useful integrals for comparison of improper integrals, on the first page of the exam (after this cover page). Any integrals or comparisons other than these should be explicitly derived.

Problem	Points	Score
1	16	
2	14	
3	10	
4	10	
5	12	
6	12	
7	14	
8	12	
Total	100	

You may find the following partial table of integrals to be useful:

$$\int e^{ax}\,\sin(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}(a\sin(bx) - b\cos(bx) + C),$$
$$\int e^{ax}\,\cos(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}(a\cos(bx) + b\sin(bx) + C),$$
$$\int \sin(ax)\sin(bx)\,dx = \frac{1}{b^2-a^2}\,(a\cos(ax)\sin(bx) - b\sin(ax)\cos(bx)) + C, a \neq b,$$
$$\int \cos(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\,(b\cos(ax)\sin(bx) - a\sin(ax)\cos(bx)) + C, a \neq b,$$
$$\int \sin(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\,(b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)) + C, a \neq b.$$
$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan(\frac{x}{a}) + C, a \neq 0$$
$$\int \frac{bx+c}{x^2+a^2}\,dx = \frac{b}{2}\ln(x^2+a^2) + \frac{c}{a}\arctan(\frac{x}{a}) + C, a \neq 0$$
$$\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{a-b}(\ln|x-a| - \ln|x-b|) + C, a \neq b$$
$$\int \frac{cx+d}{(x-a)(x-b)}\,dx = \frac{1}{a-b}((ac+d)\ln|x-a| - (bc+d)\ln|x-b|) + C, a \neq b$$
$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin(\frac{x}{a}) + C$$
$$\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \ln|x + \sqrt{x^2 \pm a^2}| + C$$

You may use the convergence properties of the following integrals:

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converges for } p > 1 \text{ and diverges for } p \leq 1,$$
$$\int_0^1 \frac{1}{x^p}\,dx \text{ converges for } p < 1 \text{ and diverges for } p \geq 1,$$
$$\int_0^\infty e^{-ax}\,dx \text{ converges for } a > 0.$$

1. [16 points] For this problem, $\int_1^5 g(x)\,dx = 12$, and $f(x) = 2x - 9$. Values of $g(x)$ are given in the table below.

x	1	2	3	4	5
$g(x)$	0.1	1.5	2	5	10

(a) [5 points of 16] Find $\int_5^7 g(f(x))\,dx$

Solution:
We use substitution with $w = f(x) = 2x - 9$. Then $w'(x) = 2$, so $dx = \frac{1}{2}dw$, and, noting that $w(5) = 1$ and $w(7) = 5$, $\int_5^7 g(f(x))\,dx = \int_1^5 \frac{1}{2} \cdot g(w)\,dw = \frac{1}{2}(12) = 6$.

(b) [5 points of 16] Find $\int_1^5 f(x) \cdot g'(x)\,dx$.

Solution:
We integrate by parts using $u = f(x) = 2x - 9$ and $v' = g'$. Then $u' = 2$ and $v = g$, so that

$$\int_1^5 f(x) \cdot g'(x)\,dx = f(x) \cdot g(x)\Big|_1^5 - \int_1^5 2g(x)\,dx = (1)g(5) + 7g(1) - 2(12)$$
$$= (10) + 7(0.1) - 24 = -13.3.$$

(c) [6 points of 16] Find $\int_1^5 \frac{g'(x)}{g(x)(g(x)+1)}\,dx$.

Solution:
First, substitute $w = g(x)$. Then $w' = g'(x)$, $w(1) = g(1) = 0.1$ and $w(5) = g(5) = 10$, so we get $\int_1^5 \frac{g'(x)}{g(x)(g(x)+1)}\,dx = \int_{0.1}^{10} \frac{1}{w(w+1)}\,dw$. We can find this by using the partial table of integrals or with partial fractions. From the table (the 8th equation) with $a = 0$ and $b = -1$, $\int_{0.1}^{10} \frac{1}{w(w+1)}\,dw = (\ln|w| - \ln|w+1|)\Big|_{0.1}^{10} = \ln(10) - \ln(0.1) - \ln(11) + \ln(1.1) = \ln(10) \approx 2.3$.
With partial fractions, $\frac{1}{w(w+1)} = \frac{A}{w} + \frac{B}{w+1}$ requires that $(A+B)w = 0$ and $A = 1$, so $B = -1$. This gives the result above.

2. [14 points] Consider the integral $\int_0^2 \frac{(\sin\sqrt{x})+1}{\sqrt{x}}\,dx$.

(a) [2 points of 14] Explain why this is an improper integral.

Solution:
We note that at $x = 0$ the denominator of the integrand is undefined, because of the factor of $\sqrt{x}$. Thus the integrand becomes infinite as $x \to 0$, and the integral is accordingly improper.

(b) [6 points of 14] Carefully show, using an appropriate comparison function (that is, without actually evaluating the integral), that this integral converges.

Solution:
We note that $0 < \sin\sqrt{x} + 1 < 2$ for $0 \le x \le 2$. Thus $\frac{\sin\sqrt{x}+1}{\sqrt{x}} < \frac{2}{\sqrt{x}}$ and $\int_0^2 \frac{\sin\sqrt{x}+1}{\sqrt{x}}\,dx < \int_0^2 \frac{2}{\sqrt{x}}\,dx = 2\int_0^2 \frac{1}{\sqrt{x}}\,dx = 2\int_0^1 \frac{1}{\sqrt{x}}\,dx + 2\int_1^2 \frac{1}{\sqrt{x}}\,dx$. We know that the last expression converges because it is the sum of an integral we know converges and a proper integral; as it is greater than our original integral, the original integral must similarly converge to a value less than the sum of the last two integrals.

(c) [6 points of 14] Carefully work out the actual value of the integral.

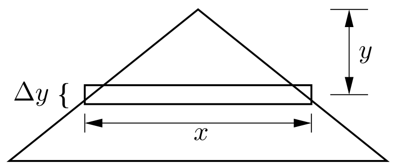
Solution:
To find the value of the integral we substitute $w = \sqrt{x}$, so that

$$\begin{aligned}\int_0^2 \frac{\sin\sqrt{x}+1}{\sqrt{x}}\,dx &= \lim_{a\to 0^+}\int_a^2 \frac{\sin\sqrt{x}+1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^{\sqrt{2}} 2\sin w + 2\,dw \\ &= \lim_{a\to 0^+}\left(-2\cos\sqrt{2} + 2\cos\sqrt{a} + 2\sqrt{2} - 2\sqrt{a}\right) = -2\cos\sqrt{2} + 2 + 2\sqrt{2}.\end{aligned}$$

3. [10 points] The Great Pyramid of Giza in Egypt was originally (approximately) 480 ft high. Its base was originally (again, approximately) a square with side lengths 760 ft.

(a) [6 points of 10] Sketch a slice that could be used to calculate the volume of the pyramid by integrating. In your sketch, indicate all variables you are using. Find an expression for the volume of the slice in terms of those variables.

Solution:
The pyramid is shown in cross-section to the right. A representative slice is given a distance y down from the top of the pyramid. It is a square box with side length x and height Δy, as shown. We can find x in terms of y by using similar triangles: the base is 760 ft and height 480 ft, so $\frac{x}{y} = \frac{760}{480}$, or $x = \frac{760}{480}y$. Thus the volume of the slice is $V_{sl} = (\text{length})^2\Delta y = (\frac{760}{480}y)^2\Delta y$ ft^3.



Alternately, we might take y to measure up from the base. In this case $\frac{x}{480-y} = \frac{760}{480}$ so that $V_{sl} = (760 - \frac{760}{480}y)^2\Delta y$ ft^3.

(b) [4 points of 10] Use your slice from (a) to find the volume of the pyramid.

Solution:
We let $\Delta y \to 0$, getting the integral $V = \int_0^{480} (\frac{760}{480}y)^2\,dy$. Expanding the square and integrating, we get

$$V = \left(\frac{760}{480}\right)^2 \int_0^{480} y^2\,dy = \frac{361}{144}\left(\frac{1}{3}\,y^3\right)\Bigg|_0^{480} = \frac{361}{432}(480^3 - 0) = 92,416,000 \text{ ft}^3.$$

Or, if we had $V_{sl} = (760 - \frac{760}{480}y)^2\Delta y$,

$$V = \int_0^{480} (760 - \frac{760}{480}\,y)^2\,dy = -(\frac{480}{760})(\frac{1}{3})(760 - \frac{760}{480})^3\Bigg|_0^{480}$$
$$= -(\frac{160}{760})(0 - 760^3) = 92,416,000 \text{ ft}^3.$$

4. [10 points] The Belgian scholar Lambert Quetelet published a distribution of chest measurements of Scottish soldiers in 1846. His distribution showed that the expected probability of a soldier having a chest measurement between 38 and 40 inches was given approximately by

$$P = \frac{1}{\sqrt{2\pi}} \int_{-0.9}^{0.1} e^{-y^2/2}\, dy.$$

Suppose that a scholar studying 19th century Scottish soldiers' physical measurements estimates this probability using the following calculation:

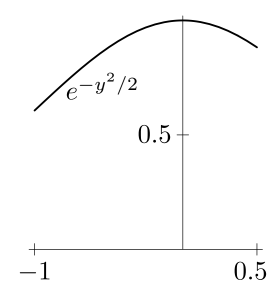
$$P = \frac{1}{\sqrt{2\pi}} \int_{-0.9}^{0.1} e^{-y^2/2}\, dy \approx \frac{1}{\sqrt{2\pi}}(0.25)(0.7406 + 0.8713 + 0.9629 + 0.9997) = 0.3565.$$

(a) [6 points of 10] What numerical method did the scholar use, how many steps were used in the method, and what was the step size?

Solution:
We know that there are four steps, because there are four terms in the sum given in the calculation. The step size is therefore $h = 0.25$, which we can also see from the factor of this h in the approximation. Then by trial and error we find that $e^{-(-0.775)^2/2} = e^{-(-0.9+0.125)^2/2} = 0.7406$, $e^{-(-0.525)^2/2} = e^{-(-0.9+0.375)^2/2} = 0.8713$, and so forth, so that the scholar has used the midpoint of each of the intervals $-9 \le x \le -0.65$, $-0.65 \le x \le -0.4$, etc., to calculate a "height" of the area approximating each interval. Thus the midpoint rule must have been used.

(b) [4 points of 10] Sketch a graph of $e^{-y^2/2}$. Explain how your graph indicates whether the scholar's approximation is an over- or under-estimate for the actual value of the integral.

Solution:
A graph of $e^{y^2/2}$ is shown to the right. This is concave down on the domain $-0.9 < y < 0.1$, so the midpoint rule, which can be visualized as summing boxes capped by the tangent to the curve, will be an overestimate for the actual value.



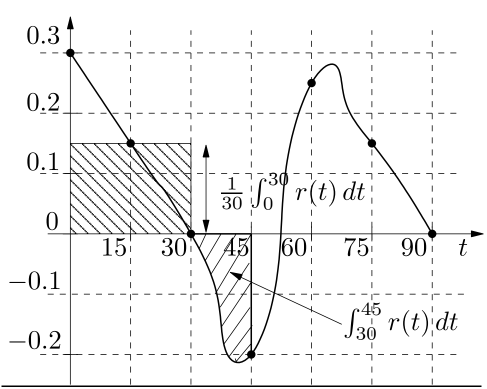
5. [12 points] Suppose that a student's rate of exam completion, given in problems per minute, is given by

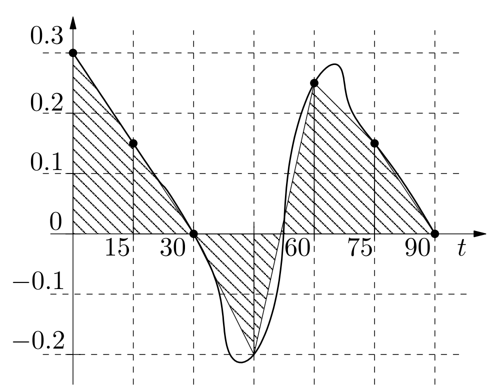
t	0	15	30	45	60	75	90
$r(t)$	0.30	0.15	0	−0.20	0.25	0.15	0

(a) [6 points of 12] Write an integral that gives the total number of problems that the student completes in the 90 minute exam period. Estimate the number of problems that the student completes. Based on your estimate, does the student complete the exam?

Solution:
Because this is a rate of completion, the total number of problems completed is $N = \int_0^{90} r(t)\,dt$. We can estimate this with a left- or right-hand sum, and a better estimate is probably the average of these. A left-hand sum gives $N \approx (15)(0.30 + 0.15 + 0 - 0.20 + 0.25 + 0.15) = 9.75$, and a right-hand sum $N \approx (15)(0.15 + 0 - 0.2 + 0.25 + 0.15 + 0) = 5.25$. Averaging these, we estimate that the total number of problems completed is $\frac{1}{2}(9.75 + 5.25) = 7.5$. The problem was supposed to say that there were eight problems on the exam, but that somehow got deleted at some point. So it's *really* hard to tell if the student finished the exam. If there are eight problems on the exam, then the student might or might not have done so.

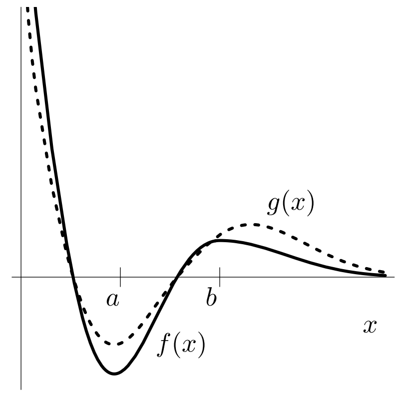
(b) [6 points of 12] The points on $r(t)$ given in (a) are shown in the two graphs below, connected by a smooth curve. On the first graph, illustrate the geometric meaning of (i) $\int_{30}^{45} r(t)\,dt$ and (ii) $\frac{1}{30}\int_0^{30} r(t)\,dt$. On the second graph, illustrate the geometric meaning of your calculation in (a).





Solution:
The solutions are graphed above. The integral $\int_{30}^{45} r(t)\,dt$ is the area between the curve and the x-axis; $\frac{1}{30}\int_0^{30} r(t)\,dt$ is the average height of the rate function for $0 \le t \le 30$, which is 0.15 (more accurately, it is the height for which the shaded box has the same area as the area under the curve). Our estimate from (a) is the trapezoid rule, which is shown on the graph to the right.

6. [12 points] Consider the graphs of $f(x)$ (the solid curve) and $g(x)$ (the dashed curve) in the figure to the right. Suppose that $\int_0^a f(x)\,dx$ converges, and that $\int_b^\infty f(x)\,dx$ diverges. Assume that the behavior suggested as $x \to 0$ and $x \to \infty$ continues for that part of the range $0 \le x < \infty$ not shown in the graph.



For each of the following indicate what this graph suggests about the convergence of the indicated integral. Circle one answer only for each part. No explanation is necessary.

(a) [2 points of 12] $\int_0^b g(x)\,dx$

converges diverges its convergence cannot be determined

Solution:
From their intersection point at the x axis to $x = 0$, $g(x) < f(x)$, so, because $\int_0^a f(x)\,dx$ converges (and because the area to the right of the x-intercept between the curves and the axis is finite), we know $\int_0^a g(x)\,dx$ must also. Obviously, $\int_a^b g(x)\,dx$ is finite, so $\int_0^b g(x)\,dx$ must converge.

(b) [2 points of 12] $\int_a^\infty g(x)\,dx$

converges diverges its convergence cannot be determined

Solution:
Similarly, this must diverge.

(c) [2 points of 12] $\int_0^\infty g(x)\,dx$

converges diverges its convergence cannot be determined

Solution:
Diverges.

(d) [2 points of 12] $\int_a^b g(x)\,dx$

converges diverges its convergence cannot be determined

Solution:
Converges. This isn't an improper integral, and the area "under" the graph is clearly finite.

(e) [2 points of 12] $\int_0^\infty (f(x) + g(x))\,dx$

converges diverges its convergence cannot be determined

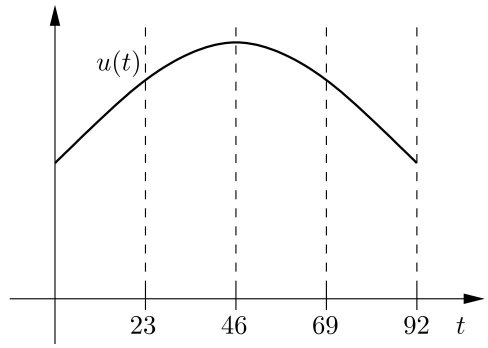
Solution:
Diverges.

(f) [2 points of 12] $\int_0^\infty (f(x) - g(x))\,dx$

converges diverges its convergence cannot be determined

Solution:
This cannot be determined.

7. [14 points] Suppose that the rate of electricity use, $u(t)$, by a large calculus-problem producing factory is given, in megawatt-hours per day for the three months of summer, by the graph to the right. Time t is given in days from the beginning of the summer. As shown, the function $u(t)$ is symmetric about the middle of summer, July 16, which falls 46 days from the beginning of the summer. Let $M(T)$ be the *average rate of electricity use* of the factory over the course of the first T days of the summer.



(a) [6 points of 14] Write a formula for $M(T)$ in terms of $u(t)$.

Solution:
We know that the average value of a function $f(x)$ over an interval $a \le x \le b$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$, so the average energy consumption over the first T days is $M(T) = \frac{1}{T}\int_0^T u(t)\,dt$.

(b) [8 points of 14] Fill in the missing blanks of the following table of values of $M(T)$. Show your work, so that it is clear how you obtained your answers.

T	23	46	69	92
$M(T)$	8			11

Solution:
The average electricity use for $0 \le t \le 23$ is 8, so, by symmetry, the average use for $69 \le t \le 92$ must also be 8. Let the average use for $23 \le t \le 46$ (or, again because of symmetry, $46 \le t \le 69$) be x. Then, if the average for the entire range $0 \le t \le 92$ is to be 11, we must have $(2x + 16)/4 = 11$, so $x = 14$. Then the values $M(46) = \frac{1}{2}(8 + 14) = 11$ and $M(69) = \frac{1}{3}(8 + 28) = 12$.

Alternately, we know that $M(23) = 8$, so $\frac{1}{23}\int_0^{23} u(t)\,dt = 8$. Thus $\int_0^{23} u(t)\,dt = 184 = \int_{69}^{92} u(t)\,dt$. Then $M(92) = 11 = \frac{1}{92}\int_0^{92} u(t)\,dt$, so $\int_0^{92} u(t)\,dt = 1012$. But $\int_0^{92} u(t)\,dt = \int_0^{23} u(t)\,dt + \int_{23}^{69} u(t)\,dt + \int_{69}^{92} u(t)\,dt = 184 + \int_{23}^{69} u(t)\,dt + 184$. Thus $\int_{23}^{69} u(t)\,dt = 1012 - 368 = 644$, so $\int_{23}^{46} u(t)\,dt = \int_{46}^{69} u(t)\,dt = 322$. Thus $M(46) = \frac{1}{46}\int_0^{46} u(t)\,dt = \frac{1}{46}(184 + 322) = 11$, and $M(69) = \frac{1}{69}\int_0^{69} u(t)\,dt = \frac{1}{69}(\int_0^{23} u(t)\,dt + \int_{23}^{69} u(t)\,dt) = \frac{1}{69}(184 + 644) = 12$.

It is also worth noting that we can deduce $M(46) = 11$ from the symmetry of the graph: the average for the two halves of the data must be equal.

8. [12 points] The velocity of an object, with air resistance, may in some circumstances be given as

$$v(t) = \sqrt{\frac{g}{k}}\left(\frac{e^{2mt}}{e^{2mt}+1} - \frac{1}{e^{2mt}+1}\right),$$

where g is the acceleration due to gravity, k is a constant representing air resistance, and $m = \sqrt{gk}$.

(a) [2 points of 12] Write an expression for the distance D that the object falls in the first t_0 seconds.

Solution:
$D = \int_0^{t_0} v(t)\,dt = \sqrt{\frac{g}{k}}\int_0^{t_0}\left(\frac{e^{2mt}}{e^{2mt}+1} - \frac{1}{e^{2mt}+1}\right)dt.$

(b) [5 points of 12] Find the distance D *(note that half of this calculation is significantly harder than the rest; do not waste too much time on it if you get stuck).*

Solution:
We can integrate both parts of the integral with the substitutions $w = e^{2mt}$, so that $\frac{1}{2m}dw = e^{2mt}dt$, or, equivalently, $dt = \frac{1}{2me^{2mt}}dw = \frac{1}{2mw}dw$. Then

$$\begin{aligned}
D = \sqrt{\frac{g}{k}}\int_0^{t_0}\left(\frac{e^{2mt}}{e^{2mt}+1} - \frac{1}{e^{2mt}+1}\right)dt &= \frac{1}{2m}\sqrt{\frac{g}{k}}\left(\int_1^{e^{2mt_0}}\frac{1}{w+1}\,dw - \int_1^{e^{2mt_0}}\frac{1}{w(w+1)}\,dw\right)\\
&= \frac{1}{2m}\sqrt{\frac{g}{k}}\left(\ln(e^{2mt_0}+1) - \ln(2) - (\ln|w| - \ln|w+1|)\Big|_1^{e^{2mt_0}}\right)\\
&= \frac{1}{2m}\sqrt{\frac{g}{k}}\left(\ln(e^{2mt_0}+1) - \ln(2) - \ln(e^{2mt_0}) + \ln(e^{2mt_0}+1) - \ln(2)\right)\\
&= \sqrt{\frac{g}{k}}\left(\frac{\ln(e^{2mt_0}+1)-\ln(2)}{m} - t_0\right),
\end{aligned}$$

where we used partial fractions (or a table) to find the second integral.

(c) [5 points of 12] Suppose $\sqrt{g/k} = 10$ and $m = 1$. Note that in this case $v(3) = 9.95 \approx 10$. Use a geometric argument to show that the distance traveled between $t = 0$ and $t = 3$, $D(3)$, satisfies the inequality $15 < D(3) < 30$.

Solution:
A graph of $v(t)$ is shown to the right, along with the graph of $v = 10$ and $v = \frac{10}{3}t$ (for $0 \le t \le 3$). Clearly the actual distance traveled (the area under $v(t)$) is between the area under $v(t) = 10$ and that under $v(t) = \frac{10}{3}t$. These areas are, respectively, $d_1 = (10)(3) = 30$ and $d_2 = \frac{1}{2}(3)(10) = 15$. Thus $15 < D(3) < 30$.