Math 116 — Second Exam

November 14, 2007

Name: Exam Solutions

Instructor: ____________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. There is a partial table of integrals, and useful integrals for comparison of improper integrals and series, on the first page of the exam (after this cover page).

Problem	Points	Score
1	6	
2	6	
3	16	
4	12	
5	16	
6	12	
7	9	
8	8	
9	15	
Total	100	

You may find the following partial table of integrals to be useful:

$$\int e^{ax}\sin(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}(a\sin(bx) - b\cos(bx) + C),$$
$$\int e^{ax}\cos(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}(a\cos(bx) + b\sin(bx) + C),$$
$$\int \sin(ax)\sin(bx)\,dx = \frac{1}{b^2-a^2}\,(a\cos(ax)\sin(bx) - b\sin(ax)\cos(bx)) + C, a \neq b,$$
$$\int \cos(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\,(b\cos(ax)\sin(bx) - a\sin(ax)\cos(bx)) + C, a \neq b,$$
$$\int \sin(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\,(b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)) + C, a \neq b.$$
$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan(\frac{x}{a}) + C, a \neq 0$$
$$\int \frac{bx+c}{x^2+a^2}\,dx = \frac{b}{2}\ln(x^2+a^2) + \frac{c}{a}\arctan(\frac{x}{a}) + C, a \neq 0$$
$$\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{a-b}(\ln|x-a| - \ln|x-b|) + C, a \neq b$$
$$\int \frac{cx+d}{(x-a)(x-b)}\,dx = \frac{1}{a-b}((ac+d)\ln|x-a| - (bc+d)\ln|x-b|) + C, a \neq b$$
$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin(\frac{x}{a}) + C$$
$$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln|x + \sqrt{x^2 \pm a^2}| + C$$

You may use the convergence properties of the following integrals and series:

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converges for } p > 1 \text{ and diverges for } p \leq 1,$$
$$\int_0^1 \frac{1}{x^p}\,dx \text{ converges for } p < 1 \text{ and diverges for } p \geq 1,$$
$$\int_0^\infty e^{-ax}\,dx \text{ converges for } a > 0,$$
$$\sum_{n=1}^\infty \frac{1}{n^p} \text{ converges for } p > 1 \text{ and diverges for } p \leq 1,$$
$$\sum_{n=1}^\infty \frac{1}{a^n} \text{ converges for } a > 1 \text{ and diverges for } a \leq 1.$$

1. [6 points] Use the integral test to determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n^5)}$$

converges or diverges. Explain your reasoning.

Solution:
The integral test says that if $f(n) > 0$ and $f'(n) < 0$, then the series $\sum_{n=a}^{\infty} f(n)$ converges or diverges $\int_c^{\infty} f(n)\,dn$ converges of diverges. Here, $f(n) = \frac{1}{n\ln(n^5)}$ is positive and decreasing, so we can investigate the convergence of the integral. Noting that $\ln(n^5) = 5\ln(n)$ and using the substitution $w = \ln(n)$, we have

$$\int_c^{\infty} \frac{1}{n\ln(n^5)}\,dn = \int_c^{\infty} \frac{1}{5n\ln(n)}\,dn = \frac{1}{5}\int_{\ln(c)}^{\infty} \frac{1}{w}\,dw,$$

which we know is divergent. Thus the series must diverge as well. Note that if we hadn't remembered that $\ln(n^5) = 5\ln(n)$, we could take $w = \ln(n^5)$ so that $w' = \frac{5n^4}{n^5}$ and $\frac{1}{5}dw = \frac{1}{n}dn$. Thus, we get the integral $\int_{\ln(c^5)}^{\infty} \frac{1}{5w\ln(w)}\,dw$, which leads to the same conclusion as before.

2. [6 points] Use one of the ratio or limit comparison tests to determine whether the series

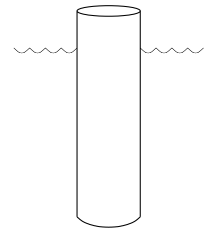
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3^n - 2}}$$

converges. Explain your reasoning.

Solution:
With the ratio test, taking $a_n = |a_n| = \frac{1}{\sqrt{3^n-2}}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\sqrt{3^n-2}}{\sqrt{3^{n+1}-2}} = \frac{\sqrt{3^n}\sqrt{1-\frac{2}{3^n}}}{\sqrt{3^n}\sqrt{3-\frac{2}{3^n}}}$. Thus, canceling the factor of $\sqrt{3^n}$, we have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\sqrt{1-\frac{2}{3^n}}}{\sqrt{3-\frac{2}{3^n}}} = \frac{1}{\sqrt{3}}$. This is less than one, so by the ratio test the series must converge.

Alternately, to use the limit comparison test, we need a good comparison series. For large n, $a_n = \frac{1}{\sqrt{3^n-2}}$ looks like $b_n = \frac{1}{\sqrt{3^n}} = \frac{1}{3^{n/2}}$, and we know that $\sum_{n=1}^{\infty} b_n$ converges (because it's a geometric series with $x < 1$). Using this for our comparison, we have $\frac{a_n}{b_n} = \frac{\sqrt{3^n}}{\sqrt{3^n-2}}$, so that, by factoring out $\sqrt{3^n}$ as above, as $n \to \infty$ we have $\frac{a_n}{b_n} \to 1$, and therefore $\sum a_n$ must also converge.

3. [16 points] A cylindrical buoy with a radius of 0.2 m and height 2 m floats with 20% of its height above water, as suggested by the figure to the right. If the density of water is $\delta = 1000$ kg/m^3 and the acceleration due to gravity is 9.81 m/s^2, find the force on the exterior of the buoy due to the hydrostatic pressure of the water.



Solution:
The hydrostatic pressure of the water at a depth h m below the surface is $\delta g h = 9810h$ N/m^2. The force on an area ΔA at that depth is therefore $9810h\Delta A$ N. The force on the bottom of the buoy is easy: the entire area is at the same depth, 1.6 m, so the force is $F_b = (9810)(1.6)(\pi(0.2)^2) = 627.84\pi \approx 1972.4$ N. Then at a depth h, a slice of height Δh around the surface of the buoy has an area $2\pi r\Delta h = 2\pi(0.2)\Delta h$ m^2, so the total force on the cylindrical exterior of the buoy is

$$\int_0^{1.6} 0.4\pi(9810)h\,dh = (0.2)\pi(9810)h^2\Big|_0^{1.6} = 1962\pi(1.6)^2 \approx 15779 \text{ N}.$$

The total force on the exterior of the buoy is therefore approximately $1972.4 + 15779 \approx 17751$ N.

4. [12 points] Zeno's paradox says that one can never arrive somewhere because one must always first travel half-way there—and that having traveled half-way there, one must travel half of the remaining distance, then half of the distance remaining after that, etc.

(a) [4 points of 12] Suppose that you start with the goal of traveling 20 km. Let d_n be the total distance that you have gone after having traveled the nth half-distance to your goal. Find d_1, d_2, d_3 and d_4.

Solution:
We start with 20 km to go, so after the first half-distance we've traveled $d_1 = 10$ km. After the second half-distance, we've traveled $d_2 = d_1 + 5 = 15$ km. After the third, we've traveled $d_3 = d_2 + \frac{5}{2} = 17.5$ km, and after the fourth, $d_4 = d_3 + \frac{5}{4} = 18.75$ km.

(b) [6 points of 12] Find a closed-form expression for the distance you've traveled after n half-distances.

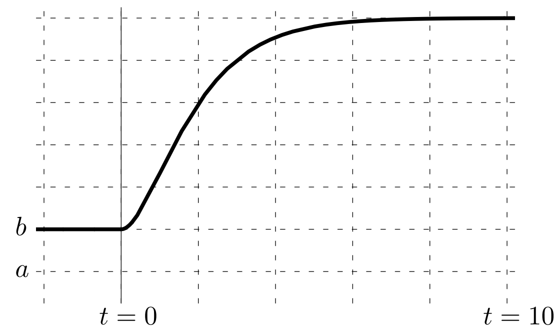
Solution:
After n half-distances, we've traveled

$$\begin{aligned} d_n &= 10 + 5 + \frac{5}{2} + \frac{5}{4} + \cdots + \frac{10}{2^{n-1}} \text{ km.} \\ &= 10(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}) \text{ km} \\ &= 10\left(\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}\right) \text{ km.} \end{aligned}$$

(c) [2 points of 12] What is the sum as the number of half-distances traveled goes to infinity? (That is, how far do you travel if you continue "forever"?)

Solution:
This is just the limit of the preceding as $n \to \infty$. In this case $\frac{1}{2^n} \to 0$, so $d \to 20$. That is, we cover the full 20 km distance.

5. [16 points] For all parts of this problem, refer to the graph to the right, which gives a cumulative distribution function $P(t)$ for some density function $p(x)$. The given graph shows all important features of the distribution (for values of t greater and less than those shown, the behavior shown continues).



(a) [4 points of 16] What are the y-values a and b? Why?

Solution:
We know that the smallest value of the cdf is zero, so $b = 0$. Similarly, the largest value is one, so $a = -1/5$.

(b) [4 points of 16] What is the approximate value of the median of this distribution?

Solution:
The median is where the cdf has the value $\frac{1}{2}$. This occurs at $t \approx 1.75$.

(c) [4 points of 16] Suppose that two points on the graph are $(3.9, 0.90)$ and $(4.1, 0.92)$. Estimate $p(4)$.

Solution:
The density function $p(x)$ is just the derivative of the cumulative distribution function $P(t)$, so we expect $p(4) \approx \frac{0.92-0.9}{4.1-3.9} = 0.10$. Alternately, we know that $\int_{3.9}^{4.1} p(x)\,dx = P(4.1) - P(3.9) = 0.02$, so because $\int_{3.9}^{4.1} p(x)\,dx \approx p(4)\cdot(0.2)$, we have $p(4)\cdot(0.2) \approx 0.02$, and therefore $p(4) \approx 0.10$.

(d) [4 points of 16] Continue to suppose that two points on the graph are $(3.9, 0.90)$ and $(4.1, 0.92)$. Estimate $\int_0^4 p(x)\,dx$.

Solution:
Again, we know that $\int_0^4 p(x)\,dx = P(4)$, so $\int_0^4 p(x)\,dx \approx \frac{1}{2}(0.92 + 0.90) = 0.91$.

6. [12 points] Recall that the Great Pyramid of Giza was (originally) approximately 480 ft high, with a square base approximately 760 ft to a side. The pyramid was made of close to 2.4 million limestone blocks, and has several chambers and halls that extended into its center. It is not too far from the truth to suppose that these open areas are located along the vertical centerline of the pyramid, and that we can therefore think of the density of the pyramid varying only along its vertical dimension. Suppose that the result is that the density of the pyramid is approximately $\delta(h) = (0.00011(h-240)^2 + 134.2)$ lb/ft^3, where h is the height measured up from the base of the pyramid.

(a) [6 points of 12] Set up an integral to find the weight W of the pyramid. You need not evaluate the integral to find the actual weight.

Solution:
Because the density of the pyramid varies along its vertical dimension, we have to slice it in that direction to find the weight of each slice and then sum them with an integral. If z is the distance down from the top of the pyramid and x is the length of the edge of the square slice, the volume of the slice is $\Delta V = x^2 \Delta z$. From similar triangles, we have that $\frac{x}{z} = \frac{760}{480}$, so that $x = \frac{760}{480} z$, so that $\Delta V = (\frac{760}{480})^2 z^2 \Delta z$. By symmetry we can see that $\delta(h) = \delta(z)$, or we can find this explicitly: $\delta(h) = \delta(480 - z) = 0.00011(480 - z - 240)^2 + 134.2 = 0.00011(240 - z)^2 + 134.2 = 0.0001(z - 240)^2 + 134.2 = \delta(z)$. Thus the weight of the pyramid is

$$W = \int_0^{480} \left(\frac{760}{480}\right)^2 z^2 (0.0001(z-240)^2 + 134.2)\, dz.$$

Evaluating this numerically, we can find $W \approx 1.2615 \times 10^{10}$ lbs (about 12.6 billion pounds).

Alternately, in terms of h, the distance up from the ground, $\Delta V = (760 - \frac{760}{480} h)^2 \Delta h$, so that $W = \int_0^{480} (760 - \frac{760}{480} h)^2 (0.00011(h-240)^2 + 134.2)\, dh$.

(b) [6 points of 12] Give an expression, in terms of integral(s), that tells how far off the ground the center of mass of the pyramid is. Again, you need not evaluate the integral(s). *(Note that you may set up the expression in terms of the weight density without worrying about converting it to a mass density.)*

Solution:
Again working with z, the distance down from the top of the pyramid, we have

$$\overline{z} = \frac{\int_0^{480} \left(\frac{760}{480}\right)^2 z^2 \delta(z) \cdot z\, dz}{W} = \frac{\int_0^{480} \left(\frac{760}{480}\right)^2 z^3 (0.0001(z-240)^2 + 134.2)\, dz}{W},$$

where W is given in part (a). Then the height of the center of mass above the surface of the desert is $\overline{h} = 480 - \overline{z}$. Using $W = 1.2615 \times 10^{10}$ lbs, found in (a), we can find $\overline{z}$ and $\overline{h}$ by evaluating the integrals numerically: $\overline{z} \approx 361$ ft, so $\overline{h} \approx 480 - \overline{z} = 119$ feet above the ground.

7. [9 points] Suppose that you invest \$5000 in a savings account that pays 2.5% interest, compounded annually. At the end of each year you withdraw the interest made on the principal in the account, and then reinvest \$100. Find a formula for R_n, the return (the amount that you take take home, after the reinvestment) from the account at the end of nth year.

Solution:
At the end of the first year the interest is $\$(0.025)(5000) = \$125$. Of that, we reinvest \$100, so the return is $R_1 = \$25$. At the end of the first year, the principal in the account is \$5100. Then, at the end of the second year the interest is $\$(0.025)(5100) = \$127.50$. We reinvest \$100, so $R_2 = \$27.50$. At the end of the second year, the principal in the account is \$5200. Thus, in the nth year, we will be getting interest on a principal of $\$(5000 + 100(n-1))$, and reinvesting \$100 of that. Thus $R_n = \$((0.025)(5000 + 100(n-1)) - 100) = \$(22.5 + 2.5n)$.

8. [8 points] Consider the area whose boundary is given in polar coordinates by the equations $\theta = \frac{\pi}{3}$ and $r = f(\theta)$. The continuous function $f(\theta)$ is defined for $\pi/3 \le \theta \le 3\pi/2$, and values of this function (spaced $\Delta\theta = 7\pi/24$ apart) are given in the table below.

$\theta =$	$\pi/3$	$5\pi/8$	$11\pi/12$	$29\pi/24$	$3\pi/2$
$f(\theta) =$	1.866	1.924	1.249	0.3912	0

Give a reasonably accurate estimate of the area of this region.

Solution:
The easiest way to estimate the area is with an integral in polar coordinates: $A = \int_{\pi/3}^{3\pi/2} \frac{1}{2} r^2\, d\theta = \int_{\pi/3}^{3\pi/2} f(\theta)^2\, d\theta$. Let's estimate this integral with the midpoint rule and $\Delta\theta = 7\pi/12$. Then

$$A = \int_{\pi/3}^{3\pi/2} \frac{1}{2} f(\theta)^2\, d\theta \approx (\frac{7\pi}{24})(1.924^2 + 0.3912^2) = 3.394.$$

We could, of course work this out with other sums; the left-hand and right-hand sums with $\Delta\theta = 7\pi/24$ are left $= (7\pi/24)(\frac{1}{2})(1.866^2 + 1.924^2 + 1.249^2 + 0.3912^2) = 4.076$ and right $= (7\pi/24)(\frac{1}{2})(1.924^2 + 1.249^2 + 0.3912^2 + 0^2) = 2.481$ Then trap $= \frac{1}{2}(4.076 + 2.481) = 3.279$.

9. [15 points] Suppose that we know that $\sum_{n=1}^{\infty} a_n$ converges*, and that* $|a_{n+1}| < |a_n|$—but we don't know what a_n is. For each of the series below, determine whether it converges, diverges, or we cannot tell (that is, there could be one value for a_n that would converge and one that would not). Circle your answer and provide a short but careful explanation for your answer (how do we know the series converges or diverges?, or, what examples show that we cannot tell?). *(The italicized instruction was omitted from the printed version of the exam.)*

(a) [3 points of 15] $\sum_{n=1}^{\infty} |a_n|$ converges diverges cannot tell

Solution:
We cannot tell if this converges or not. For example, if $a_n = (-1)^n/n$ then the original series converges, but this series does not. But if $a_n = 1/n^2$, then both converge.

(b) [3 points of 15] $\sum_{n=1}^{\infty} (-1)^n |a_n|$ converges diverges cannot tell

Solution:
Because we know that $|a_{n+1}| < |a_n|$ and $\lim_{n\to\infty} a_n = 0$ (because $\sum a_n$ converges), we have convergence by the alternating series test.

(c) [3 points of 15] $\sum_{n=1}^{\infty} \frac{a_n+1}{a_n+5}$ converges diverges cannot tell

Solution:
This diverges. We know that $\lim_{n\to\infty} a_n = 0$, so the terms of this series do not go to zero: $\lim_{n\to\infty} \frac{a_n+1}{a_n+5} = \frac{1}{5}$. Thus this series must diverge.

(d) [3 points of 15] $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converges diverges cannot tell

Solution:
This converges. Clearly $|\frac{a_n}{n^2}| < \frac{1}{n^2}$ for large enough n, so we know that $\sum \frac{a_n}{n^2}$ converges absolutely.

(e) [3 points of 15] $\sum_{n=1}^{\infty} \frac{3^n a_n}{n^3}$ converges diverges cannot tell

Solution:
We cannot tell if this converges or not. For example, if $a_n = 1/n^2$, this series is $\sum \frac{3^n}{n^5}$, which diverges because exponentials dominate power functions and the terms in the series therefore do not go to zero. However, if $a_n = 1/3^n$, it clearly converges.