Math 116 — Final Exam

December 17, 2007

Name: Exam Solutions

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. There is a partial table of integrals and series, and useful integrals for comparison of improper integrals and series, on the first page of the exam (after this cover page).

Problem	Points	Score
1	7	
2	7	
3	8	
4	12	
5	12	
6	14	
7	10	
8	10	
9	12	
10	8	
Total	100	

You may find the following partial table of integrals to be useful:

$$\int e^{ax}\sin(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}(a\sin(bx)-b\cos(bx))+C,$$
$$\int e^{ax}\cos(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}(a\cos(bx)+b\sin(bx))+C,$$
$$\int \sin(ax)\sin(bx)\,dx = \frac{1}{b^2-a^2}\,(a\cos(ax)\sin(bx)-b\sin(ax)\cos(bx))+C, a\neq b,$$
$$\int \cos(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\,(b\cos(ax)\sin(bx)-a\sin(ax)\cos(bx))+C, a\neq b,$$
$$\int \sin(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\,(b\sin(ax)\sin(bx)+a\cos(ax)\cos(bx))+C, a\neq b.$$
$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan(\frac{x}{a})+C, a\neq 0$$
$$\int \frac{bx+c}{x^2+a^2}\,dx = \frac{b}{2}\ln(x^2+a^2)+\frac{c}{a}\arctan(\frac{x}{a})+C, a\neq 0$$
$$\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{a-b}(\ln|x-a|-\ln|x-b|)+C, a\neq b$$
$$\int \frac{cx+d}{(x-a)(x-b)}\,dx = \frac{1}{a-b}((ac+d)\ln|x-a|-(bc+d)\ln|x-b|)+C, a\neq b$$
$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin(\frac{x}{a})+C$$
$$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln|x+\sqrt{x^2\pm a^2}|+C$$

You may use the convergence properties of the following integrals and series:

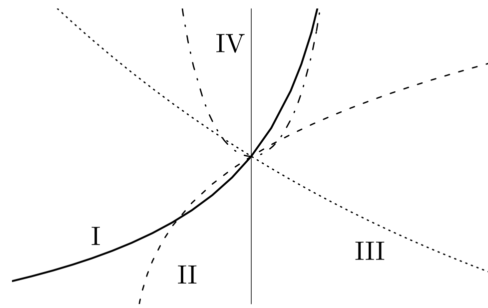
$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converges for } p>1 \text{ and diverges for } p\leq 1,$$
$$\int_0^1 \frac{1}{x^p}\,dx \text{ converges for } p<1 \text{ and diverges for } p\geq 1,$$
$$\int_0^\infty e^{-ax}\,dx \text{ converges for } a>0,$$
$$\sum_{n=1}^\infty \frac{1}{n^p} \text{ converges for } p>1 \text{ and diverges for } p\leq 1,$$
$$\sum_{n=1}^\infty \frac{1}{a^n} \text{ converges for } a>1 \text{ and diverges for } a\leq 1.$$

You may assume that the following series are known:

$$\sin(x) = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots,$$
$$\cos(x) = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots,$$
$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots,$$
$$(1+x)^p = 1+px+\frac{p(p-1)}{2!}x^2+\frac{p(p-1)(p-2)}{3!}x^3+\cdots$$

And it may be useful to note that the acceleration due to gravity is approximately $g = 9.8$ m/s^2, or $g = 32$ ft/s^2. The density of water is $\delta = 1$ kg/liter (1 mg/ml), or $\delta = 62.4$ lb/ft^3 (the latter being a density in terms of *weight*, not *mass*).

1. [7 points] The figure to the right shows four functions graphed near $x = 0$. Use Taylor series to determine which of these is the graph of $f(x) = \frac{1}{\sqrt{1-x}}$. Explain how you know which it is. *Note: you will get zero credit if you do not include work that shows how you obtained your answer using Taylor series.*



Solution:
Using the binomial series, we have

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots,$$

so near zero our function has a positive slope and is concave up. The only of the four functions shown that matches this behavior is (I).

2. [7 points] Suppose that we apply Euler's method with $\Delta x = 0.1$ to approximate the solution to $y' = ay + cx$ with $y(0) = 5$, and find that $y(0.1) \approx 6$ and $y(0.2) \approx 7.23$. What are a and c?

Solution:
Using Euler's method, we know

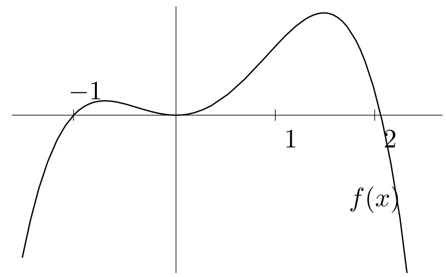
$$y(0.1) \approx y(0) + 0.1(a(y(0) + c(0))) = 5 + 0.1(a(5)) = 5 + 0.5a.$$

We found $y(0.1) \approx 6$, so $a = 2$. Then

$$y(0.2) \approx y(0.1) + 0.1(2y(0.1) + c(0.1)) = 6 + 0.1(2(6) + c(0.1)) = 7.2 + 0.01c.$$

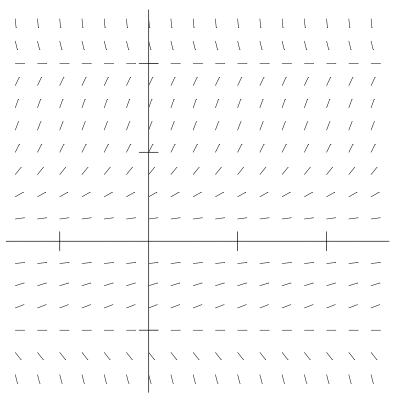
This must equal 7.23, so $c = 3$.

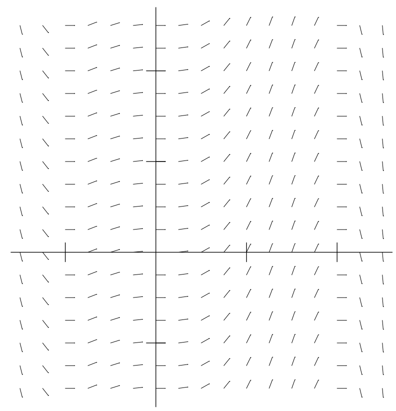
3. [8 points] Suppose that $\frac{dy}{dx} = f(x)$, where $f(x)$ is shown in the graph to the right.

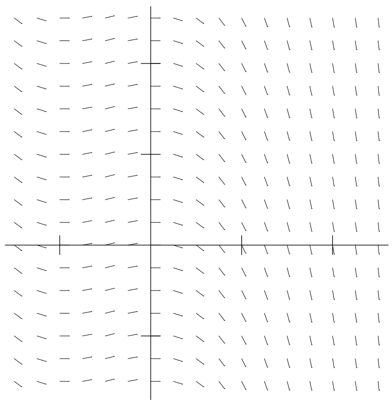


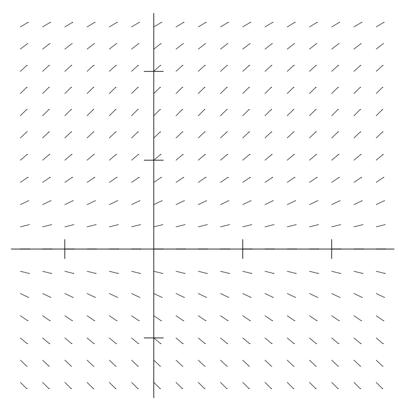
(a) [4 points of 8] Which of the slope fields below (which have ticks with unit spacing) could be the slope field of this differential equation? Explain briefly.

Solution:
The equation matches slope field (B): the slope varies only with x, so it must match (B) or (C), and must be zero at $x = -1$, $x = 0$ and $x \approx 2.1$, so it must be (B).

A.


B.


C.


D.


(b) [4 points of 8] Are there any equilibrium solutions to this differential equation? If so, what are they? If not, why not?

Solution:
There are no equilibrium solutions (which are $y =$ constant). Because $\frac{dy}{dx} = f(x)$ (and $f(x)$ is not everywhere zero), there is no way that $y =$ constant can solve the differential equation.

4. [12 points] Newton's law of cooling (or warming) says that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium. Suppose that a thermometer used by a veterinarian to find the temperature of a sick horse obeys Newton's law of cooling. Further suppose that before insertion the thermometer reads 82° F, after one minute it reads 92° F, and after another minute it reads 97° F, and that a sudden convulsion unexpectedly destroys the thermometer after the 97° reading. Call the horse's temperature T_h.

(a) [3 points of 12] Write a differential equation for the temperature T (a function of time t) of the thermometer. Your equation may involve the constant T_h.

Solution:
We know that $\frac{dT}{dt}$ is proportional to the temperature difference between thermometer and horse, so

$$\frac{dT}{dt} = k(T_h - T),$$

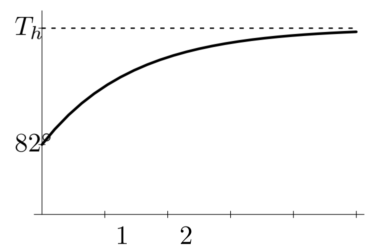
where T_h is the horse's temperature.

(b) [3 points of 12] Solve the differential equation for T to find a general solution for T. Your solution may include undetermined constants such as T_h.

Solution:
Separating variables, we have $\frac{dT}{T-T_h} = -kdt$, so that $\ln|T - T_h| = -kt + A$. Exponentiating and setting $C = \pm e^A$, we find $T = T_h + Ce^{-kt}$.

(c) [3 points of 12] Sketch a graph of T, indicating the initial temperature and T_h on your graph.

Solution:
We know that T is increasing, is an exponential function, and that it decays towards T_h. Further, we know that the time scale should be in minutes because of the initial data given. This leads to the graph shown to the right.



(d) [3 points of 12] Write a set of equations that would allow you to determine the horse's temperature (and the other undetermined constants in your expression for T). *Do not solve these equations.*

Solution:
When $t = 0$, $T = 82$, so $T = T_h + C = 82$; when $t = 1$, $T = T_h + Ce^{-k} = 92$; and when $t = 2$, $T = T_h + Ce^{-2k} = 97$. This gives us a set of equations for T_h, k and C,

$$\begin{aligned} T_h + C &= 82 \\ T_h + Ce^{-k} &= 92 \\ T_h + Ce^{-2k} &= 97, \end{aligned}$$

that would allow us to solve for T_h.

5. [12 points] Solve the following:

(a) [4 points of 12] Explain how, by starting with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \qquad |x| < 1,$$

you can derive the Taylor series for $\ln(1+x)$, $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$. (Your explanation need not be a step-by-step derivation, but should clearly indicate what steps are necessary to complete it.)

Solution:
Substituting $-x$ for x in the geometric series, we obtain

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots,$$

so that, integrating term-by-term, we have

$$\ln(1+x) = C + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

We know that $\ln(1+0) = \ln(0) = 0$, so $C = 0$.

(b) [4 points of 12] If money is invested at an interest rate r, compounded monthly, it will double in n years, where n is given by

$$n = \frac{\ln(2)}{12}\frac{1}{\ln(1+\frac{r}{12})}.$$

(This can be deduced from the formula $(1+\frac{r}{12})^{12n} = 2$, but we do not need this derivation for this problem.) Use the Taylor polynomial of degree 1 for $\ln(1+x)$ near $x = 0$ to show that for small r the doubling time n is approximately proportional to $\frac{1}{r}$, and find the constant of proportionality, k.

Solution:
The Taylor polynomial of degree one, $P_1(x)$, is just the Taylor series truncated at the linear term: $P_1(x) = x$. Thus $\ln(1+\frac{r}{12}) \approx \frac{r}{12}$ for small r, and

$$n = \frac{\ln(2)}{12}\frac{1}{\ln(1+\frac{r}{12})} \approx \frac{\ln(2)}{12}\frac{1}{\frac{r}{12}} = \frac{\ln(2)}{r}.$$

We see that n is proportional to $\frac{1}{r}$, and the constant of proportionality is $k = \ln(2)$.

(c) [4 points of 12] Use the Taylor polynomial of degree 2 for $\ln(1+x)$ near $x = 0$ to show that for small r the doubling time $n = \frac{\ln(2)}{12}\frac{1}{\ln(1+\frac{r}{12})}$ may be approximated by an expression of the form $\frac{k}{r-ar^2}$. Find k and a.

Solution:
The Taylor polynomial of degree 2, $P_2(x)$, is just the Taylor series truncated at the quadratic term: $P_2(x) = x - \frac{1}{2}x^2$. Thus $\ln(1+\frac{r}{12}) \approx \frac{r}{12} - \frac{1}{2}(\frac{r}{12})^2$, so

$$n = \frac{\ln(2)}{12}\frac{1}{\ln(1+\frac{r}{12})} \approx \frac{\ln(2)}{12}\frac{1}{\frac{r}{12} - \frac{1}{2}(\frac{r}{12})^2} = \frac{\ln(2)}{r - \frac{1}{24}r^2}.$$

We therefore have $k = \ln(2)$, as we found before, and $a = \frac{1}{24}$.

6. [14 points] Consider the power series
$$\sum_{n=3}^{\infty} \frac{(3-x)^{3n}}{8^n(n-2)}.$$

(a) [5 points of 14] Does this power series converge at $x = 1$? Explain.

Solution:
No. When $x = 1$, the series becomes $\sum\limits_{n=3}^{\infty} \frac{2^{3n}}{8^n(n-2)} = \sum\limits_{n=3}^{\infty} \frac{1}{(n-2)}$. Then $\frac{1}{n-2} > \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges, so this must diverge also.

(b) [5 points of 14] Does this power series converge at $x = 5$? Explain.

Solution:
Yes. When $x = 5$, the series becomes $\sum\limits_{n=3}^{\infty} \frac{(-2)^{3n}}{8^n(n-2)} = \sum\limits_{n=3}^{\infty} \frac{(-1)^{3n}}{(n-2)}$. The terms of this series alternate sign, are decreasing in magnitude, and $\lim\limits_{n\to\infty} \frac{1}{n-2} = 0$, so by the alternating series test the series converges.

(c) [4 points of 14] Find the interval of convergence of this power series. Be sure it is clear how you find your answer.

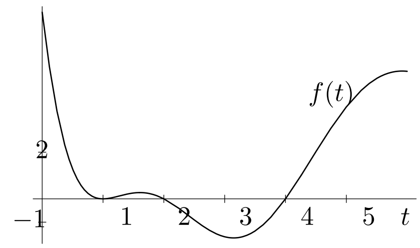
Solution:
To find the radius of convergence we use the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(3-x)^{3n+3}}{8^{n+1}(n-1)} \cdot \frac{8^n(n-2)}{(3-x)^{3n}}\right| = |(3-x)^3| \lim_{n\to\infty} \frac{n-2}{8(n-1)} = \frac{1}{8}|(3-x)^3|.$$

For convergence, we know that this ratio must be less than one, so $|x-3|^3 < 8$, or $|x-3| < 2$, and the radius of convergence is 2. Thus the interval of convergence is at least $1 < x < 5$, and, from the work in (a) and (b), we know that it in fact includes $x = 5$ but not $x = 1$. The interval of convergence is $1 < x \le 5$.

Alternately, we know that the series is centered on $x = 3$. At $x = 1$ and $x = 5$ (a distance of two from $x = 3$) it diverges and converges, respectively. Thus the interval of convergence is $1 < x \le 5$.

7. [10 points] Define a function F for $x \geq 0$ by $F(x) = \int_x^{2x} f(t)\,dt$, where $f(t)$ is given by the graph to the right.



(a) [4 points of 10] Find $F'(1)$ (show your work).

Solution:
We know that $F(x) = \int_x^a f(t)\,dt + \int_a^{2x} f(t)\,dt = -\int_a^x f(t)\,dt + \int_a^{2x} f(t)\,dt$, for some a between x and $2x$. Thus, from the construction theorem (second fundamental theorem of calculus), $F'(x) = -f(x) + 2f(2x)$, and we therefore have $F'(1) = -f(1) + 2f(2) = 0$.

(b) [6 points of 10] If the second degree Taylor polynomial for $F(x)$ near $x = 1$ is $P_2(x) = a + b(x-1) + c(x-1)^2$, what is b? What is the sign of a? The sign of c? Why?

Solution:
We know that $a = F(1)$, $b = F'(1)$ and $c = \frac{1}{2}F''(1)$. Thus, from (a), we know that $b = 0$. Then $a = \int_1^2 f(t)\,dt$. The area under $f(t)$ is above the t-axis between $t = 1$ and $t = 2$, so $a > 0$. Finally, $F''(x) = \frac{d}{dx}(-f(x) + 2f(2x)) = -f'(x) + 4f'(2x)$, so $F''(1) = -f'(1) + 4f'(2) = 0 + 4f'(2)$. At $x = 2$, the slope of f is negative, so $F''(1) < 0$ and similarly $c < 0$.

8. [10 points] The function $f(x) = \dfrac{c}{1+x^2}$ $(-\infty < x < \infty)$ occurs in probability theory as the density function of the Cauchy distribution.

(a) [5 points of 10] Find c (show your work).

Solution:
For this to be a density function, we know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. This $f(x)$ is symmetric about $x = 0$, so this is equivalent to taking half of the integral to be $1/2$.

$$\int_0^{\infty} \frac{c}{1+x^2}\,dx = \lim_{b\to\infty} c\cdot\arctan(b) - c\cdot\arctan(0) = \lim_{b\to\infty} c\cdot\arctan(b) = \frac{\pi}{2}c.$$

Thus $\frac{\pi}{2}c = \frac{1}{2}$, and $c = \frac{1}{\pi}(\approx 0.3183)$.

(b) [5 points of 10] Electrostatic charge is distributed along an infinite straight rod according to the Cauchy distribution. What proportion of the charge is located more than one unit of length away from the origin?

Solution:
This is equal to the area under $f(x)$ for $|x| > 1$, or, equivalently, $1 - \int_{-1}^{1} f(x)\,dx$. Thus

$$\text{proportion} = 1 - \frac{1}{\pi}\int_{-1}^{1} \frac{1}{1+x^2}\,dx = 1 - \frac{2}{\pi}\int_0^1 \frac{1}{1+x^2}\,dx = 1 - \frac{2}{\pi}\arctan(1) = 1 - \frac{2}{\pi}\cdot\frac{\pi}{4} = \frac{1}{2}.$$

9. [12 points] A leaky bucket of water is carried from the desert floor to the top of the Great Pyramid[1], about 150 m above the desert floor. Suppose that we can ignore the mass of the bucket by comparison to the mass of the water, and that the mass of the water in the bucket is given as $m(t)$ kg, where t is given in minutes. The ascent from the ground to the top of the pyramid is made at a constant speed and takes 15 min ($t = 0$ corresponds to the beginning of the ascent).

(a) [6 points of 12] Write an integral that gives the work done to carry the bucket from the desert floor to the top of the pyramid. Be sure that it is clear how you obtained your integral.

Solution:
Consider the work done between a time t and a time $t+\Delta t$. The distance traveled is $d = 10\Delta t$, and the force required is $m(t) \cdot g$. Thus the work for this small section of time is $\Delta W = m(t) \cdot g \cdot 10\Delta t$, and the total work is just

$$W = \int_0^{15} 10g \cdot m(t)\, dt.$$

(Where g is the acceleration due to gravity, about 9.8 m/s^2.)

Alternately, consider the work done to lift the bucket from a height h to a height $h + \Delta h$. The distance traveled is Δh m, and the force required is $m(t_h) \cdot g$, where t_h is the time at which we are at height h. If it takes 15 min to climb the pyramid, the (constant) velocity is $v = 150$ m/15 min $=$ 10 m/min. So the height as a function of time is $h(t) = 10 \cdot t$ m, and $t_h = h/10$. Thus the work is to move the bucket the height Δh is $\Delta W = m(h/10) \cdot g \cdot \Delta h$, and the total work is

$$W = \int_0^{150} m(\frac{h}{10}) \cdot g\, dh.$$

(This can be obtained from the previous integral with the substitution $t = h/10$.)

(b) [6 points of 12] Suppose that $m(t)$ is given in the table below. Use (all of) this data to estimate the total work to carry the bucket to the top of the pyramid.

$t =$ (in min)	0	3	6	9	12	15
$m(t) =$ (in kg)	4	3.5	3	2.7	2.3	2

Solution:
Using the second integral above and a left-hand sum with $\Delta t = 3$ min, we have

$$W \approx (98)(3)(4 + 3.5 + 3 + 2.7 + 2.3) = 4557 \text{ N} \cdot \text{m}.$$

A right-hand sum gives a similarly accurate result, 3969 N·m, and the trapezoid rule ($= \frac{1}{2}(4557 + 3969) = 4263$ N·m) would be significantly more accurate.

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[1] Everyone together now: "No, not *again*!"

10. [8 points] Each of the following statement is either False (there are counter-examples to the statement), True, or True if a condition holds. For each, circle the correct characterization (obviously, a True statement is also True if the condition holds; circle "True" in this case, not "True if. . . "). No explanations are necessary.

(a) [2 points of 8] $y = 3x^2$ is a solution to $xy' = 2y - b$

True False $\boxed{\text{True if } b = 0}$

Solution:
True if $b = 0$: $y' = 6x$, so $xy' = 6x^2 = 2(3x^2) = 2y$.

(b) [2 points of 8] $\int_{-1}^{1} \frac{1}{1+kx^2}\,dx$ is an improper integral.

True False $\boxed{\text{True if } k \le -1}$

Solution:
True if $k \le -1$: If $k > -1$ there is no singularity in the denominator and the integral is proper.

(c) [2 points of 8] If $F'(x) = x\sin(e^x)$, then $F(x) = \int_0^\infty t\sin(e^t)\,dt$.

True $\boxed{\text{False}}$ True if $F(0) = 0$

Solution:
False: if $F'(x) = x\sin(e^x)$ we can write $F(x) = \int_c^x t\sin(e^t)\,dt$, and if we also know $F(0) = 0$, then $F(x) = \int_0^x t\sin(e^t)\,dt$. But neither of these is the same as $\int_0^\infty t\sin(e^t)\,dt$.

(d) [2 points of 8] $F(t) = \begin{cases} 0, & t < 0 \\ t/a, & 0 \le t < a \\ 1, & t \ge a \end{cases}$ could be a cumulative distribution function.

$\boxed{\text{True}}$ False True if $a = 1$

Solution:
True: $F(t) = 0$ at the left end of its domain and 1 at the right end, and is never decreasing.