Math 116 — First Midterm

October 14, 2008

Name: ______________________________

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	10	
2	12	
3	14	
4	16	
5	6	
6	6	
7	12	
8	12	
9	12	
Total	100	

1. [10 points] A model[1] for the extinction rate of marine species during the Phanerozoic period (which extends from approximately 545 million years before the present until now) stipulates that this extinction rate, $r(t)$, in numbers of marine animal families per million years, is

$$r(t) = \frac{3130}{t + 262},$$

where t is the number of million years after the start of the Phanerozoic period.

a. [5 points] Find an expression for $E(t)$, the number of extinctions that occurred between the start of the Phanerozoic period and t million years thereafter.

b. [5 points] Find an expression for the average rate of extinctions between the start of the Phanerozoic period and t million years thereafter.

[1]Newman & Eble, *Decline in Extinction Rates and Scale Invariance in the Fossil Record*, Paleobiology **25**:234–39 (1999)

2. [12 points] Let $f(x)$ be a positive, continuous and differentiable, non-constant function. Let $F(x)$ be an antiderivative of f that passes through the origin. For each of the following, find all values of the constant a for which the statement is true. Include your work and/or a short explanation so that it is clear how you obtain your answers.

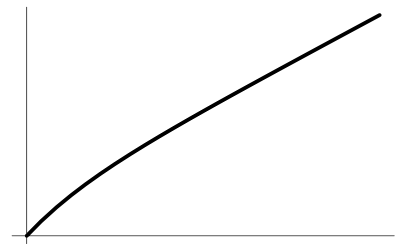
a. [4 points] $F(x) = \int_a^x f(t)\,dt$

b. [4 points] $\int_0^a x\,f'(x)\,dx = f(a) - F(a)$.

c. [4 points] $\int 5\,f(\frac{x}{a})\,dx = F(\frac{x}{a}) + C$

3. [14 points] An astute University of Michigan squirrel notes that the length of one strand of the web of a mathematically inclined spider is exactly $\int_0^6 \sqrt{2 + 2e^{-x} + e^{-2x}}\, dx$ cm, where x measures the horizontal distance from the wall of a campus building. The strand of web is shown in the figure to the right, below.

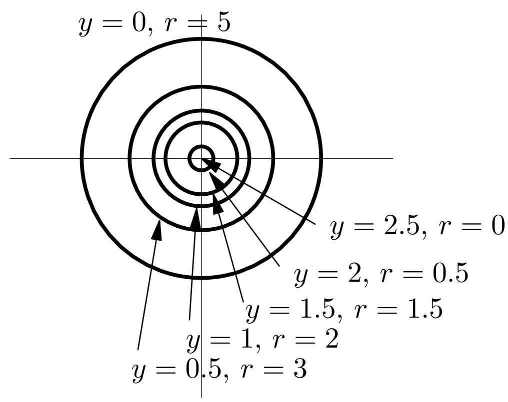
a. [7 points] Find an equation $y = f(x)$ that describes the shape of this strand of web.



b. [7 points] Estimate the length of the strand of web using MID(3). Is your estimate an over- or underestimate? How do you know?

4. [16 points] A University of Michigan squirrel, in Peru for a study-abroad semester, discovers a singularly symmetric pond hidden high in the Andes mountains. The pond has perfectly circular horizontal cross-sections, and its radii r at different depths y are shown (in meters) in the figure to the right, below. As shown, the outer edge of the pond has a radius of 5 meters, and the pond gets deeper towards its center.

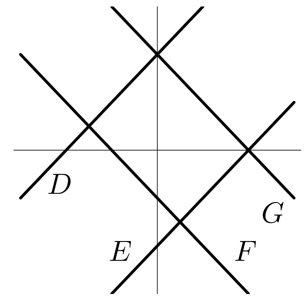
a. [5 points] Set up an integral that gives the total volume of the pond. Your integral may involve the radius (r) and/or depth (y) of the pond. Be sure it is clear how you obtain your answer.



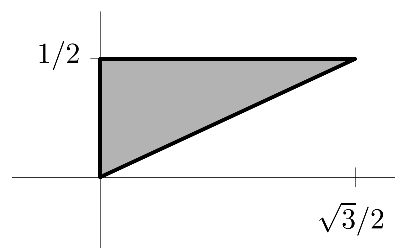
b. [5 points] Estimate the volume of the pond based on your work in (a).

c. [6 points] The pond is fed by a stream that is drying up as time goes on. If the stream delivers water to the pond at a rate of $r(t) = 60t\,e^{-t^2}$ m^3/year, does the pond ever fill? (Assume that the pond starts out empty when $t = 0$, and ignore other effects such as evaporation and rainfall.)

5. [6 points] Consider a parametric curve given by $x(t) = f(t)$, $y(t) = g(t)$, where $f(5) = 0$, $g(5) = 3$, $f'(5) > 0$ and $g'(5) < 0$. Which of the lines D, E, F, or G in the figure below could be the line tangent to the curve $(x(t), y(t))$ at $t = 5$? Explain.



6. [6 points] Find a set of inequalities in polar coordinates that describe the shaded triangle in the figure shown to the right, below.



7. [12 points] Each of the following integrals is improper. For each, carefully determine its convergence or divergence by using the comparison test. Be sure to indicate what you are using for your comparison and all steps that allow you to conclude convergence or divergence of the given integral. Mathematical precision is important in this problem.

a. [6 points] $\displaystyle\int_1^\infty \frac{3+\sin\phi}{\phi}\,d\phi$

b. [6 points] $\displaystyle\int_0^1 \frac{dz}{\sqrt{z^3+z}}$

8. [12 points] Some values of the continuous, differentiable function $g(x)$ are given in the table below.

x	1	5/4	3/2	7/4	2
$g(x)$	2	3	4	7	10

a. [6 points] Estimate the integral $\int_1^4 \frac{g(\sqrt{t})}{\sqrt{t}}\,dt$ using these data.

b. [6 points] Estimate the integral $\int_1^4 g(\sqrt{t}\,)\,dt$ using these data.

9. [12 points] For the following, $f(x) > g(x) > 0$ and a is a positive constant. Indicate if each is true or false by circling **True** or **False**. For each, include **one sentence** to explain your answer.

a. [3 points] If $\int_1^\infty f(x)\,dx$ converges, then $\int_1^\infty f(a+x)\,dx$ must converge.

True False

b. [3 points] If $\int_1^\infty f(x)\,dx$ converges, then $\int_1^\infty (a+f(x))\,dx$ must converge.

True False

c. [3 points] If $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ both converge, then $\int_1^\infty f(x)\cdot g(x)\,dx$ must converge.

True False

d. [3 points] If $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ both converge, then $\int_1^\infty \frac{f(x)}{g(x)}\,dx$ must converge.

True False