Math 116 — First Midterm

October 14, 2008

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	10	
2	12	
3	14	
4	16	
5	6	
6	6	
7	12	
8	12	
9	12	
Total	100	

1. [10 points] A model[1] for the extinction rate of marine species during the Phanerozoic period (which extends from approximately 545 million years before the present until now) stipulates that this extinction rate, $r(t)$, in numbers of marine animal families per million years, is

$$r(t) = \frac{3130}{t + 262},$$

where t is the number of million years after the start of the Phanerozoic period.

a. [5 points] Find an expression for $E(t)$, the number of extinctions that occurred between the start of the Phanerozoic period and t million years thereafter.

Solution: Using the fundamental theorem of calculus, this is just

$$E(t) = \int_0^t \frac{3130}{x + 262}\,dx = \left(3130 \ln|x + 262|\right)\Big|_0^t = 3130\ln(t + 262) - 3130\ln(262).$$

(units are marine animal families).

b. [5 points] Find an expression for the average rate of extinctions between the start of the Phanerozoic period and t million years thereafter.

Solution: The average rate is

$$A(t) = \frac{1}{t - 0}\int_0^t \frac{3130}{x + 262}\,dx = \frac{1}{t}\left(3130\ln(t + 262) - 3130\ln(262)\right).$$

(units are marine animal families/million years).

[1]Newman & Eble, *Decline in Extinction Rates and Scale Invariance in the Fossil Record*, Paleobiology **25**:234–39 (1999)

2. [12 points] Let $f(x)$ be a positive, continuous and differentiable, non-constant function. Let $F(x)$ be an antiderivative of f that passes through the origin. For each of the following, find all values of the constant a for which the statement is true. Include your work and/or a short explanation so that it is clear how you obtain your answers.

a. [4 points] $F(x) = \int_a^x f(t)\,dt$

Solution: This is true if $a = 0$. We know that $F(x)$ and $\int_a^x f(t)\,dt$ are both antiderivatives of $f(x)$. $F(x)$ passes through $(0, 0)$, while $\int_a^x f(t)\,dx$ passes through $(a, 0)$. Thus $a = 0$.

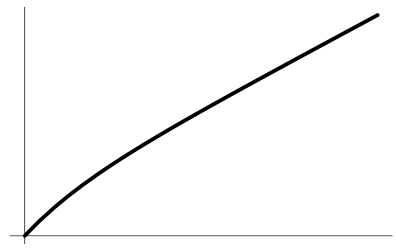
b. [4 points] $\int_0^a x\,f'(x)\,dx = f(a) - F(a)$.

Solution: This is true if $a = 1$. Using integration by parts, $\int_0^a x\,f'(x)\,dx = x\,f(x)\Big|_0^a - \int_0^a f(x)\,dx = a\,f(a) - F(a)$. For this to equal $f(a) - F(a)$, we must have $a = 1$.

c. [4 points] $\int 5\,f(\frac{x}{a})\,dx = F(\frac{x}{a}) + C$

Solution: This is true when $a = \frac{1}{5}$. Differentiating both sides of the equation, we get $5f(\frac{x}{a}) = \frac{1}{a}\,F'(\frac{x}{a}) = \frac{1}{a}\,f(\frac{x}{a})$. Thus a must equal $\frac{1}{5}$.
Alternately, integrating the left hand side, we have $\int 5\,f(\frac{x}{a})\,dx = 5aF(\frac{x}{a}) + C$. Thus for the two sides to be equal we must have $a = \frac{1}{5}$.

3. [14 points] An astute University of Michigan squirrel notes that the length of one strand of the web of a mathematically inclined spider is exactly $\int_0^6 \sqrt{2+2e^{-x}+e^{-2x}}\,dx$ cm, where x measures the horizontal distance from the wall of a campus building. The strand of web is shown in the figure to the right, below.



a. [7 points] Find an equation $y = f(x)$ that describes the shape of this strand of web.

Solution: We note that the squirrel is integrating to find the arclength of a curve. The arclength of a curve $y = f(x)$ is given by $\int_a^b \sqrt{1+(f'(x))^2}\,dx$, so we must have $(f'(x))^2 = 1 + 2e^{-x} + e^{-2x}$. Thus $f'(x) = \pm(1 + e^{-x})$. To match the graph shown we must take the positive sign, so $f(x) = x - e^{-x} + C$, for some constant C. Because the web starts at the origin $((x, y) = (0, 0))$, C must be 1, so $y = x - e^{-x} + 1$.

b. [7 points] Estimate the length of the strand of web using MID(3). Is your estimate an over- or underestimate? How do you know?

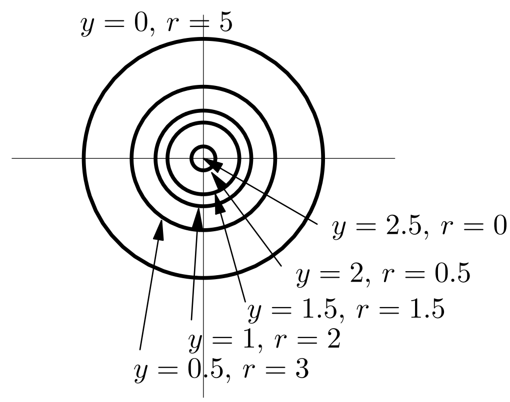
Solution: We're estimating the integral $\int_0^6 \sqrt{2+2e^{-x}+e^{-2x}}\,dx$. The three intervals being considered are $0 \le x \le 2$, $2 \le x \le 4$, and $4 \le x \le 6$, and the midpoints of these intervals are $x = 1$, $x = 3$ and 5. Thus the midpoint approximation is

$$\text{MID}(3) = 2\left(\sqrt{2+2e^{-1}+e^{-2}} + \sqrt{2+2e^{-3}+e^{-6}} + \sqrt{2+2e^{-5}+e^{-10}}\right).$$

(Which is approximately 9.127 cm.) By graphing the function $g(x)$ we can see that it is concave up, so we know that the midpoint is an underestimate for the actual length.

4. [16 points] A University of Michigan squirrel, in Peru for a study-abroad semester, discovers a singularly symmetric pond hidden high in the Andes mountains. The pond has perfectly circular horizontal cross-sections, and its radii r at different depths y are shown (in meters) in the figure to the right, below. As shown, the outer edge of the pond has a radius of 5 meters, and the pond gets deeper towards its center.

a. [5 points] Set up an integral that gives the total volume of the pond. Your integral may involve the radius (r) and/or depth (y) of the pond. Be sure it is clear how you obtain your answer.



Solution: The depth from the surface of the pond down to a "slice" of the pond is y. The radius of the pond at that depth is $r(y)$, so that the cross-sectional area of the slice at that depth is $\pi\,(r(y))^2$. Thus the volume of the slice is $\pi\,(r(y))^2\,\Delta y$, where Δy is the thickness of the slice. We can sum these up as $\Delta y \to 0$ by using an integral:

$$\text{Volume of pond} = \int_0^{2.5} \pi\,(r(y))^2\,dy.$$

(It is also possible to integrate vertical, circular shells with thickness Δr and height $y(r)$. This gives the integral $V = \int_0^5 2\pi\, r\, y(r)\, dr$.)

b. [5 points] Estimate the volume of the pond based on your work in (a).

Solution: Because the radii for the cross sections of the pond are given at intervals of $\Delta y = \frac{1}{2}$ m, we can use a left- or right-hand sum with this Δy.

$$\begin{aligned}\text{LEFT} &= \tfrac{\pi}{2}\left(r(0)^2 + r(1/2)^2 + r(1)^2 + r(3/2)^2 + r(2)^2\right)\\ &= \tfrac{\pi}{2}\left(5^2 + 3^2 + 2^2 + (3/2)^2 + (1/2)^2\right) = \tfrac{81\pi}{4} \approx 64 \text{ m}^3\\ \text{RIGHT} &= \tfrac{\pi}{2}\left(3^2 + 2^2 + (3/2)^2 + (1/2)^2 + 0^2\right) = \tfrac{31\pi}{4} \approx 24 \text{ m}^3.\end{aligned}$$

The trapezoid estimate is the average of these two, $\frac{56\pi}{4} \approx 44$ m^3.

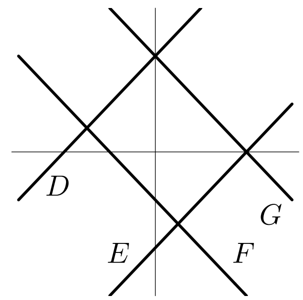
c. [6 points] The pond is fed by a stream that is drying up as time goes on. If the stream delivers water to the pond at a rate of $r(t) = 60t\,e^{-t^2}$ m^3/year, does the pond ever fill? (Assume that the pond starts out empty when $t = 0$, and ignore other effects such as evaporation and rainfall.)

Solution: The total amount of water that will flow into the pond is given by $\int_0^\infty r(t)\,dt$, which is an improper integral. We find the value of this integral by taking the appropriate limit:

$$\int_0^\infty r(t)\,dt = \lim_{b\to\infty} \int_0^b 60t\,e^{-t^2}\,dt = \lim_{b\to\infty} -30\,e^{-t^2}\Big|_0^b = \lim_{b\to\infty} -30(e^{-b^2} - 1) = 30 \text{ m}^3.$$

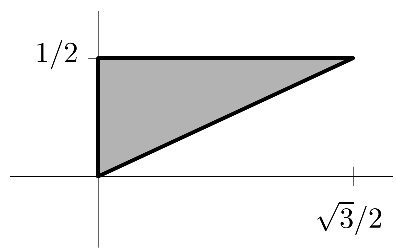
This is below the trapezoid estimate, so we would expect that the pond does not fill (though because it is between the left- and right-hand estimates, it might).

5. [6 points] Consider a parametric curve given by $x(t) = f(t)$, $y(t) = g(t)$, where $f(5) = 0$, $g(5) = 3$, $f'(5) > 0$ and $g'(5) < 0$. Which of the lines D, E, F, or G in the figure below could be the line tangent to the curve $(x(t), y(t))$ at $t = 5$? Explain.



Solution: Because $f'(5) > 0$ and $g'(5) < 0$, we know that the slope of the tangent line must be $\frac{dy}{dx} = \frac{g'(5)}{f'(5)} < 0$, so the tangent line must be one of F or G. Then we know that when $x(5) = f(5) = 0$, $y(5) = g(5) > 0$, so the y-intercept of the line must be positive, and we therefore know that the tangent must be line G.

6. [6 points] Find a set of inequalities in polar coordinates that describe the shaded triangle in the figure shown to the right, below.



Solution: The top boundary is $y = 1/2$, so on that boundary $r\sin(\theta) = 1/2$, or $r = \frac{1}{2\sin(\theta)}$. This holds for θ between $\theta = \arctan(\frac{1/2}{\sqrt{3}/2}) = \frac{\pi}{6}$ and $\theta = \frac{\pi}{2}$, so the inequalities are $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$, $0 \leq r \leq \frac{1}{2\sin(\theta)}$. We could also say $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$, $r \geq 0$, $r\sin\theta \leq \frac{1}{2}$.

It is also correct to find equations for each of the boundaries and to then convert those to polar coordinates. This gives $r\sin\theta \leq \frac{1}{2}$, $r\cos\theta \geq 0$ and $r\sin\theta \geq \frac{1}{\sqrt{3}} r\cos\theta$. These are, of course, equivalent to the preceding; the first is $r \leq \frac{1}{2\sin\theta}$, the third gives $\theta \geq \frac{\pi}{6}$, and the second requires that $r \geq 0$ and $\theta \leq \frac{\pi}{2}$ (or $r < 0$ and $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$, but these inequalities result in the same region).

7. [12 points] Each of the following integrals is improper. For each, carefully determine its convergence or divergence by using the comparison test. Be sure to indicate what you are using for your comparison and all steps that allow you to conclude convergence or divergence of the given integral. Mathematical precision is important in this problem.

a. [6 points] $\displaystyle\int_1^\infty \frac{3+\sin\phi}{\phi}\,d\phi$

Solution: A good comparison function is $g(\phi) = \frac{1}{\phi}$ (or $\frac{2}{\phi}$), because for large ϕ the integrand will have the same behavior as $\frac{1}{\phi}$. We know that $g(\phi) \le \frac{3+\sin\phi}{\phi}$, because $3+\sin(\phi) \ge 2$. Then we know that $\int_1^\infty g(\phi)\,d\phi$ diverges; thus $\int_1^\infty \frac{3+\sin\phi}{\phi}\,d\phi$ must also diverge.

b. [6 points] $\displaystyle\int_0^1 \frac{dz}{\sqrt{z^3+z}}$

Solution: A good comparison function is $g(z) = \frac{1}{z^{1/2}}$, because for small z the integrand will look like $\frac{1}{z^{1/2}}$ (z^3 is *less than* z for small values of z). We note that $g(z) > \frac{1}{\sqrt{z^3+z}}$, and know that $\int_0^1 \frac{1}{z^{1/2}}\,dz$ converges. Thus $\int_0^1 \frac{1}{\sqrt{z^3+z}}$ must also converge.

8. [12 points] Some values of the continuous, differentiable function $g(x)$ are given in the table below.

x	1	5/4	3/2	7/4	2
$g(x)$	2	3	4	7	10

a. [6 points] Estimate the integral $\int_1^4 \frac{g(\sqrt{t})}{\sqrt{t}}\,dt$ using these data.

Solution: Let $w = \sqrt{t}$. Then $dw = \frac{1}{2\sqrt{t}}\,dt = \frac{1}{2w}\,dt$, and the integral is $\int_1^4 g(\sqrt{t}\,)\,dt = \int_1^2 2\,g(w)\,dw$. We can estimate this integral from the given data: a left-hand sum is

$$\text{LHS} = \tfrac{1}{4}(2)\,(2+3+4+7) = 8.$$

We could also use a right-hand sum or trapezoid estimate:

$$\text{RHS} = \tfrac{1}{4}\,(2)\,\,(3+4+7+10) = 12,$$
$$\text{and}\quad \text{TRAP} = \tfrac{1}{2}\,(8+12) = 10.$$

b. [6 points] Estimate the integral $\int_1^4 g(\sqrt{t}\,)\,dt$ using these data.

Solution: Let $w = \sqrt{t}$. Then $dw = \frac{1}{2\sqrt{t}}\,dt = \frac{1}{2w}\,dt$, and the integral is $\int_1^4 g(\sqrt{t}\,)\,dt = \int_1^2 2w\,g(w)\,dw$. We can estimate this integral from the given data:

w	1	5/4	3/2	7/4	2
$g(w)$	2	3	4	7	10
$2w\,g(w)$	4	15/2	12	49/2	40

Thus, a left-hand sum gives $\int_1^2 2w\,g(w)\,dw \approx \frac{1}{4}(4+15/2+12+49/2) = 12$, a right-hand sum gives $\int_1^2 2w\,g(w)\,dw \approx \frac{1}{4}(15/2+12+49/2+40) = 21$, and an average of the two may be expected to be a reasonable estimate for the actual value of the integral:

$$\int_1^2 2w\,g(w)\,dw \approx \tfrac{33}{2} = 16.5.$$

9. [12 points] For the following, $f(x) > g(x) > 0$ and a is a positive constant. Indicate if each is true or false by circling **True** or **False**. For each, include **one sentence** to explain your answer.

a. [3 points] If $\int_1^\infty f(x)\,dx$ converges, then $\int_1^\infty f(a+x)\,dx$ must converge.

$\boxed{\text{True}}$ False

Solution: $\int_1^\infty f(a+x)\,dx = \int_{1+a}^\infty f(x)\,dx$, which integrates over a smaller region than the convergent integral $\int_1^\infty f(x)\,dx$.

b. [3 points] If $\int_1^\infty f(x)\,dx$ converges, then $\int_1^\infty (a+f(x))\,dx$ must converge.

True $\boxed{\text{False}}$

Solution: $a + f(x) > a$ for all x, and $\int_1^\infty a\,dx$ clearly diverges, so $\int_1^\infty (a+f(x))\,dx$ also diverges.

c. [3 points] If $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ both converge, then $\int_1^\infty f(x)\cdot g(x)\,dx$ must converge.

$\boxed{\text{True}}$ False

Solution: This turns out to be more interesting than one might expect. A reasonable expectation is that $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$, so that for large enough x, $f(x)\cdot g(x) < g(x)$ and so by comparison $\int_1^\infty f(x)\cdot g(x)\,dx$ must converge. However, if $f(x)$ and $g(x)$ do not go to zero this may be false. Therefore either True or False was accepted as a correct answer, with credit given for the explanation if it supported the answer given.

d. [3 points] If $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ both converge, then $\int_1^\infty \frac{f(x)}{g(x)}\,dx$ must converge.

True $\boxed{\text{False}}$

Solution: Because $0 < g(x) < f(x)$, $\frac{f(x)}{g(x)} > 1$, so this clearly diverges.