Math 116 — Second Midterm

November 12, 2008

Name: EXAM SOLUTIONS

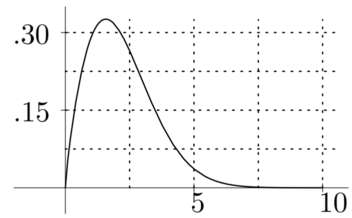
Instructor: ____________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	13	
2	4	
3	8	
4	16	
5	13	
6	15	
7	15	
8	16	
Total	100	

1. [13 points] A common model for the distribution of wind speeds, v, is the Rayleigh distribution. A graph of an example of the Rayleigh distribution is shown in the figure to the right, below.

a. [2 points] Is this a *probability density function* (PDF) or *cumulative distribution function* (CDF)? Why?



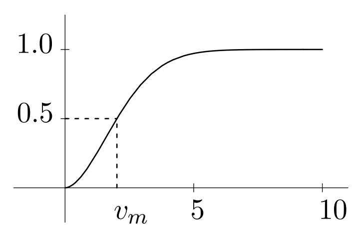
Solution: This is a *probability density function*. There are many reasons for this: a CDF must never decrease, and must end with a value of one, neither of which are true for this graph; and the area under the PDF is one, while the area under the CDF is infinite.

b. [4 points] What is the meaning of the "5" on the x-axis of this graph? Given that the function value at 5 is 0.037, what is the meaning of the expression $(0.037)(5.01 - 4.99)$?

Solution: The variable on the x-axis is wind speed (v), so the "5" on the x-axis corresponds to $v = 5$. The expression $(0.037)(5.01 - 4.99) = .00074$ is an approximation to the area under the curve between $v = 4.99$ and $v = 5.01$, so it gives the approximate fraction of wind speeds that are between 4.99 and 5.01 (that is, about 0.074%).

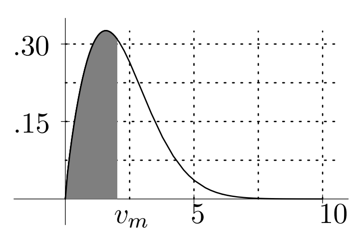
c. [3 points] If you identified the figure in (a) as a PDF, sketch the corresponding CDF; conversely, if you identified it as a CDF, sketch the corresponding PDF.

Solution: Because the given graph is a PDF, the corresponding CDF gives, at each velocity v, the proportion of the time the wind has that velocity or less. This is the antiderivative of the given PDF that passes through the origin (because the probability of seeing a wind with a negative velocity is zero), and it must end at one, as shown in the figure.



d. [4 points] Mark the *median* wind-speed on both the graph from (a) (reproduced below), and your graph in part (c). How did you locate the median?

Solution: The median windspeed v_m is the speed for which half of the observed winds have velocities less, and half more, than that speed. This is shown in the CDF by finding the velocity where the CDF has a value of one-half, as shown above. On the PDF it is the velocity at which half of the area lies to the left, as shown below.



2. [4 points] Let $a_n = \sum_{k=1}^{n} \frac{1}{k}$. Circle all of the statements given below that are true about the sequence a_n and *briefly* explain your answers.

-successive values of the sequence a_n increase in magnitude

-successive values of the sequence a_n decrease in magnitude

-successive values of the sequence a_n may increase or decrease in magnitude

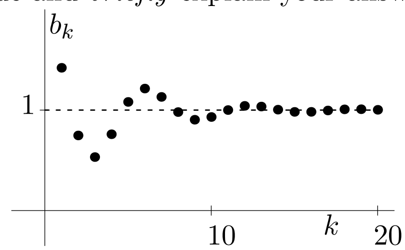
-the sequence a_n converges

-the sequence a_n diverges

-it is not possible to determine whether sequence a_n converges or diverges

Solution: For any n, we have $a_{n+1} = a_n + \frac{1}{n+1}$, so successive values of the sequence a_n increase in magnitude. Then as $n \to \infty$, $a_n \to \sum_{k=1}^{\infty} \frac{1}{k}$, which diverges.

3. [8 points] Let b_k be given by the graph below (as $k \to \infty$, the behavior shown in the graph continues as is suggested by the figure). For each part of the problem below, circle all of the given statements that are true and *briefly* explain your answers.



a. [4 points] Consider the sequence b_k.

-the sequence b_k could be defined by $b_k = 1 - \frac{(-1)^k}{k}$

-the sequence b_k can only be defined recursively

-it is impossible to find a recursive definition for the sequence b_k

-the sequence b_k converges

-the sequence b_k diverges

-it is not possible to determine whether the sequence b_k converges or diverges

Solution: Noting that the given formula gives the sequence 2, 1/2, 4/3, 3/4,... , with terms alternating above and below one, the given formula cannot be correct. The statements about the recursive definition are nonsensical in this case. By inspection of the graph, the sequence b_k converges to 1.

b. [4 points] Consider the series $\sum_{k=1}^{\infty} b_k$.

-the sequence of partial sums S_n of the series converges

-the sequence of partial sums S_n of the series diverges

-it is not possible to determine whether the sequence of partial sums S_n of the series converges or diverges

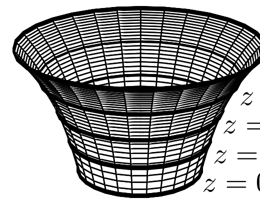
-the series $\sum_{k=1}^{\infty} b_k$ converges

-the series $\sum_{k=1}^{\infty} b_k$ diverges

-it is not possible to determine whether the series $\sum_{k=1}^{\infty} b_k$ converges or diverges

Solution: As $k \to \infty$, we see that $b_k \to 1$, not zero, so we know that the series must diverge. Because convergence of the series is by definition the convergence of the sequence of partial sums, the sequence of partial sums S_n must also diverge.

4. [16 points] An entrepreneurial University of Michigan Business Squirrel is marketing childrens' buckets with curved sides, as shown in the figure to the right, below. The figure gives the radius of the bucket, r, at different heights, z, from the bottom of the bucket. All lengths are given in inches. Suppose that a child fills one of these buckets with muddy water.



a. [4 points] If the density of the water in the bucket is $\delta(z)$ oz/in^3, write an integral that gives the mass of the water in the bucket.

Solution: Slicing the bucket horizontally, the mass is

$$M = \int_0^8 \pi\,\delta(z)\,(r(z))^2\,dz \quad \text{oz.}$$

b. [4 points] If $\delta(z) = (24 - z)$ oz/in^3, estimate the mass using your integral from (a).

Solution: The data given provide values of $r(z)$ at steps of size $\Delta z = 2$ in. We can find $\delta(z)$ at each of the points $z = 0$, 2, etc., and find a left or right Riemann sum for the mass. A better estimate would be the average of the two (the trapezoid estimate):

$$\begin{aligned}
\text{LEFT} &= 2\pi\left(24(1)^2 + 22(3/2)^2 + 20(9/4)^2 + 18(3)^2\right) \approx 2116 \text{ oz}\\
\text{RIGHT} &= 2\pi\left(22(3/2)^2 + 20(9/4)^2 + 18(3)^2 + 16(6)^2\right) \approx 5584 \text{ oz}\\
\text{TRAP} &\approx 0.5\,(2116 + 5584) = 3850 \text{ oz}
\end{aligned}$$

c. [8 points] Estimate the center of mass of the bucket.

Solution: By symmetry, the center of mass must be along the centerline of the bucket, so if $\overline{x}$ and $\overline{y}$ give the coordinates along the base of the bucket, we know $\overline{x} = 0$ and $\overline{y} = 0$. The z center of mass is

$$\overline{z} = \frac{\int_0^8 \pi\,z\,\delta(z)\,(r(z))^2\,dz}{M} \quad \text{in,}$$

where M is the mass given in (a). We have the mass from (b), and so need only estimate the integral $\int_0^8 \pi\,z\,\delta(z)\,(r(z))^2\,dz$. Again, we can find a left or right Riemann sum or a trapezoid estimate:

$$\begin{aligned}
\text{LEFT} &= 2\pi\left(24(0)(1)^2 + 22(2)(3/2)^2 + 20(4)(9/4)^2 + 18(6)(3)^2\right) = 2952\pi \approx 9274\\
\text{RIGHT} &= 2\pi\left(22(2)(3/2)^2 + 20(4)(9/4)^2 + 18(6)(3)^2 + 16(8)(6)^2\right) = 12{,}168\pi \approx 38{,}227\\
\text{TRAP} &\approx 0.5\,(2952\pi + 12{,}168\pi) = 7560\pi \approx 23{,}750.
\end{aligned}$$

Then the center of mass is $\overline{z} \approx 23{,}750/3850 = 6.17$ in. Using just a left-sum, we have $\overline{z} \approx 9274/2116 = 4.60$ in, and the right-sum gives $\overline{z} \approx 38{,}227/5584 = 6.85$ in.

5. [13 points] Suppose that when a fire alarm is set off in East Hall, the occupants (being mathematicians) leave at precise five-minute intervals. At the end of each interval, 75% of those who were in the building at the beginning of the interval exit the building. Suppose that on a sunny Friday afternoon at 2PM a fire alarm goes off when there are 400 mathematicians in East Hall.

a. [4 points] Find the number of mathematicians that leave at the end of the first, second, and nth five-minute intervals.

Solution: It is easiest to start with the general case: the number of mathematicians in the building at the beginning of the nth five-minute time interval is $400(0.25)^{n-1}$. Thus the number of mathematicians leaving at the end of the nth five minute interval is $0.75(400(0.25)^{n-1}) = 300(0.25)^{n-1}$. This means that the number leaving at the end of the first five-minute interval is 300 ($= 300(0.25)^0$), and the second, 75 ($= 300(0.25)^1$).

b. [5 points] Let $L(n)$ be the total number of mathematicians who have left East Hall at the end of the nth five-minute interval after the alarm started. Find a closed-form expression for $L(n)$.

Solution: We note that $L(n)$ is just the sum of the terms in (a):

$$L(n) = 300 + 300(0.25) + \cdots + 300(0.25)^{n-1}.$$

We note that this is just a finite geometric series with n terms and a coefficient of 300. Thu

$$L(n) = 300\left(\frac{1-0.25^n}{1-0.25}\right) = 400\,(1-0.25^n).$$

(Alternately, we could note that at the end of the nth five-minute interval the number of mathematicians left in the building is $400(0.25)^n$. Thus, the number who have left must be $400 - 400(0.25)^n = 400(1-0.25^n)$.)

c. [4 points] How many mathematicians will leave the building if the alarm goes on forever? (Justify your answer mathematically.)

Solution: As $n \to \infty$, the numerator of $L(n)$ goes to one, so $L(n) \to 300\frac{1}{1-0.25} = 400$. So all of the mathematicians actually leave!

6. [15 points] For each of the following, assume that $\sum a_n$ and $\sum b_n$ are both convergent series, and that $a_n > a_{n+1} > b_n > b_{n+1} > 0$. For each, explain your answer in a sentence or two, or with a clear picture or counterexample.

a. [3 points] Is $\sum (b_n - a_n)$ a convergent series? Explain.

Solution: Yes, $\sum (b_n - a_n)$ is a convergent series. We know $\sum a_n$ and $\sum b_n$ converge, so $\sum (b_n - a_n)$ must converge to $\sum b_n - \sum a_n$.

b. [3 points] Is $\sum (a_n \cdot b_n)$ a convergent series? Explain.

Solution: Yes, $\sum a_n \cdot b_n$ is convergent. We know that $a_n \to 0$ as $n \to \infty$, so for some sufficiently large value of n, $a_n < 1$. Thus for sufficiently large values of n, $a_n \cdot b_n < b_n$, and therefore because $\sum b_n$ converges, by the comparison test $\sum a_n \cdot b_n$ must also converge.

c. [3 points] Is $\sum ((-1)^n \ln(a_n + 1))$ a convergent series? Explain.

Solution: Yes, $\sum (-1)^n \ln(a_n + 1)$ is a convergent series. We know (because $\sum a_n$ converges) that $a_n \to 0$ as $n \to \infty$, so $\ln(a_n + 1) \to \ln(1) = 0$, and we're given that $a_n > a_{n+1} > 0$, so $\ln(a_n + 1) > \ln(a_{n+1} + 1) > 0$. Thus $\ln(a_n + 1) \to 0$ monotonically from above as $n \to \infty$, and $\sum (-1)^n \ln(a_n + 1)$ is an alternating series. Thus, by the alternating series test we know that the series converges.

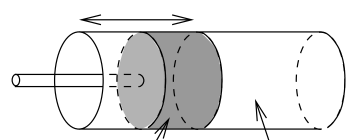
d. [3 points] Is $\sum (2\,a_n)$ a convergent series? Explain.

Solution: Yes, $\sum 2\,a_n$ is a convergent series. Because $\sum a_n$ is convergent, $2\sum a_n$ is clearly well-defined, and $2\sum a_n = \sum 2\,a_n$.

e. [3 points] Is $\sum ((-1)^n \sqrt{b_n}\,)$ an absolutely convergent series? Explain.

Solution: We can't tell if $\sum (-1)^n \sqrt{b_n}$ is absolutely convergent. This requires that $\sum \sqrt{b_n}$ converge, which we can't tell because for sufficiently large n it must be that $\sqrt{b_n} > b_n$, and thus the convergence of $\sum b_n$ doesn't tell us what happens to $\sum \sqrt{b_n}$.

7. [15 points] Consider a piston that compresses a closed cylinder of gas, as shown in the figure to the right, below. If the volume of the gas in the cylinder is V, then the force required to move the piston and compress the gas is $F = \frac{k}{V^{1.4}}$, where k is a constant. The uncompressed length of the gas cylinder is 2 ft and its radius is $\frac{1}{4}$ ft. Let x be the distance that the piston has moved to compress the gas. (Note that the volume of a cylinder with radius r and height h is $\pi r^2 h$.)



a. [5 points] Find an expression for $F(x)$, the force as a function of x. If $F(0) = 200$ lb, find k.

Solution: The volume when the gas has been compressed a distance x is $V = \pi\,(\frac{1}{4})^2\,(2-x)$, so the force as a function of x is $F(x) = \frac{k}{((\pi/16)\,(2-x))^{1.4}}$. When $x = 0$ the volume is $V = \pi\,(\frac{1}{4})^2\,2 = \frac{\pi}{8}$ ft^3, so we have $F(0) = 200 = \frac{k}{(\pi/8)^{1.4}}$, and $k = 200(\frac{\pi}{8})^{1.4} \approx 54$.

b. [10 points] Find the work to compress the gas from $x = 0$ to $x = \frac{3}{2}$.

Solution: Using $F(x)$ from above, the work to compress the gas from a distance x to $x + \Delta x$ is $\Delta W = \frac{k}{((\pi/16)\,(2-x))^{1.4}}\,\Delta x$. Thus, letting $\Delta x \to 0$, the total work to compress the gas from $x = 0$ to $x = \frac{3}{2}$ is

$$\int_0^{3/2} \frac{k}{((\pi/16)\,(2-x))^{1.4}}\,dx.$$

Integrating, we have

$$\begin{aligned}\int_0^{3/2} \frac{k}{((\pi/16)\,(2-x))^{1.4}}\,dx &= \frac{k}{0.4(\pi/16)^{1.4}}\,(2-x)^{-0.4}\Big|_0^{3/2}\\ &= \frac{k}{0.4(\pi/16)^{1.4}}\,((\frac{1}{2})^{-0.4} - 2^{-0.4}) \approx 741 \text{ ft} \cdot \text{lb}.\end{aligned}$$

8. [16 points] For each of the following series, state a convergence test that you could use to determine if the series converges or not and indicate why you chose that test. Then carefully apply the test to determine if the series converges or not. Mathematical precision is important in this problem.

a. [8 points] $\displaystyle\sum_{n=2} \frac{\sqrt{n+3}}{n^2-1}$

Solution: This looks like a good problem to try and do a comparison test, but the most obvious comparison ($\frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$) isn't larger than the given function, so we use the *Limit Comparison Test*. We know that $\sum n^{-3/2}$ converges. The limit we consider is $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$, which with $a_n = \frac{\sqrt{n+3}}{n^2-1}$ and $b_n = \frac{1}{n^{3/2}}$ is

$$\lim_{n\to\infty} \frac{\sqrt{n+3}}{n^2-1} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{n^2\sqrt{1+\frac{3}{n}}}{n^2-1} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{3}{n}}}{1-\frac{1}{n^2}} = 1.$$

This is a finite non-zero limit, so we know that the convergence properties of both of these series are the same, so the given series must converge.

b. [8 points] $\displaystyle\sum \frac{(n+1)!}{2e^{3n}}$

Solution: This problem involves factorials and exponentials, so the ratio test is a good one to use. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{(n+2)!}{(2\,e^{3n+3})!}\right)\left(\frac{2e^{3n}}{(n+1)!}\right) = \lim_{n\to\infty} \left(\frac{n+2}{e^3}\right) \to \infty.$$

This diverges, so by the ratio test the series diverges.