Math 116 — Final Exam

December 11, 2008

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. There is a short table of "known" Taylor series, integrals, and geometry formulas which you may use, without derivation, on the first page of this exam.

Problem	Points	Score
1	10	
2	10	
3	16	
4	12	
5	12	
6	12	
7	16	
8	12	
Total	100	

You may find the following expressions useful. And you may not. But you may use them if they prove useful.

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots$$

"Known" integral expressions:

$$\int x^n \ln x \, dx = \frac{1}{n+1}\, x^{n+1} \ln x - \frac{1}{(n+1)^2}\, x^{n+1} + C$$

$$\int e^{ax} \sin(bx)\, dx = \frac{1}{a^2+b^2}\, e^{ax}\, (a \sin(bx) - b\cos(bx)) + C$$

$$\int e^{ax} \cos(bx)\, dx = \frac{1}{a^2+b^2}\, e^{ax}\, (a \cos(bx) + b\sin(bx)) + C$$

$$\int \sin(ax)\sin(bx)\, dx = \frac{1}{b^2-a^2}\, (a\cos(ax)\sin(bx) - b\sin(ax)\cos(bx)) + C, \qquad a \neq b$$

$$\int \cos(ax)\cos(bx)\, dx = \frac{1}{b^2-a^2}\, (b\cos(ax)\sin(bx) - a\sin(ax)\cos(bx)) + C, \qquad a \neq b$$

$$\int \sin(ax)\cos(bx)\, dx = \frac{1}{b^2-a^2}\, (b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)) + C, \qquad a \neq b$$

"Known" equations from geometry:

Volume of a sphere: $V = \frac{4}{3}\pi r^3$

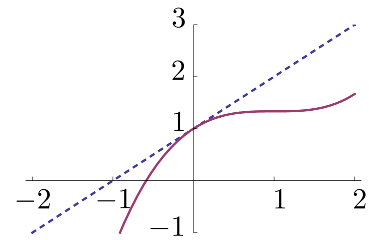
Surface area of a sphere: $A = 4\pi r^2$

Volume of a cylinder: $V = \pi r^2 h$

Volume of a cone: $V = \frac{1}{3}\pi r^2 h$

1. [10 points] Suppose that the first and third degree Taylor polynomials, $P_1(x)$ and $P_3(x)$, approximating a function $g(x)$ at $a = 0$ are given in the graph to the right, below.

a. [6 points] Using these Taylor polynomials, what are $g(0)$ and $g'(0)$? What is the sign of $g''(0)$?



Solution: We know $g(0) = 1$, and from $P_1(x)$, $g'(0) = 1$. Then $P_3(x)$ appears to be concave down, so we know that $g''(0) < 0$. We cannot, however, determine a value for $g''(0)$.

b. [4 points] Could $g(x)$ be the function $1 + \sin(x)$? Why or why not?

Solution: For this $g(x)$ we know $g(0) = 1$, $g'(0) = \cos(0) = 1$ and $g''(0) = -\sin(0) = 0$. While the first two of these are consistent with our observations in (a), the last is not, so $g(x)$ could not be the function $1 + \sin(x)$.

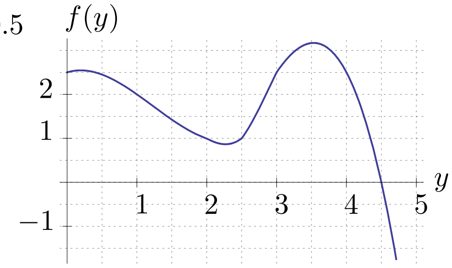
2. [10 points] Find the interval of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n\,(x-1)^n}{n}$.

Solution: To find the radius of convergence, we use the ratio test:

$$\lim_{n\to\infty}\left|\frac{2^{n+1}\,(x-1)^{n+1}}{n+1}\cdot\frac{n}{2^n\,(x-1)^n}\right| = |x-1|\lim_{n\to\infty}\frac{2n}{n+1} = 2|x-1|.$$

This must be less than one, so $|x-1|<\frac{1}{2}$, and the radius is $R=\frac{1}{2}$. Testing convergence at the endpoints, we have at $x=\frac{1}{2}$ the series $\sum \frac{2^n\,(-\frac{1}{2})^n}{n} = \sum \frac{(-1)^n}{n}$, the alternating harmonic series, which converges. At $x=\frac{3}{2}$, we similarly have $\sum \frac{2^n\,(\frac{1}{2})^n}{n} = \sum \frac{1}{n}$, which is the harmonic series, which diverges. Thus the interval of convergence is $\frac{1}{2}\le x<\frac{3}{2}$.

3. [16 points] Suppose that $\frac{dy}{dt} = f(y)$, where $f(y)$ is given by the graph in the figure to the right, below.



a. [4 points] If $y(0) = 1$, use Euler's method with $\Delta t = 0.5$ to estimate $y(1)$.

Solution: Using Euler's method, we approximate

$$y(0.5) \approx y(0) + 0.5\, f(y(0)) = 1 + 0.5(2) = 2,$$

and

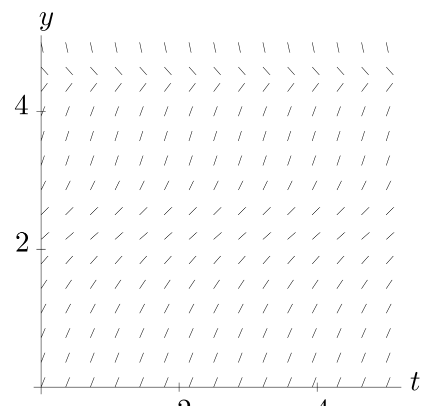
$$y(1) \approx y(0.5) + 0.5\, f(y(0.5)) = 2 + 0.5(1) = 2.5.$$

b. [4 points] Could $y(t) = 2.5 - t^2$ be a solution to the given differential equation $\frac{dy}{dt} = f(y)$? Why or why not?

Solution: No; $\frac{dy}{dt} = -2t = \mp 2\sqrt{2.5 - y}$, which clearly could not generate the given graph.
Alternately, note that if we start at $(0, 1)$ we know $\frac{dy}{dt} = f(1) = 2$, but if $y = 2.5 - t^2$, $\left.\frac{dy}{dt}\right|_{t=0} = 0 \neq 2$.

c. [4 points] Could the slope field given to the right, below, be the slope field for the given differential equation $\frac{dy}{dt} = f(y)$? Why or why not?

Solution: This could be the indicated slope field; it depends only on y, and the slopes at different y values appear to be similar to the function values $f(y)$ shown in the figure above.



d. [4 points] Are there any equilibrium solutions to the given differential equation $\frac{dy}{dt} = f(y)$? If so, are they stable? If there are none, why are there none?

Solution: Yes, $y = 4.5$ is an equilibrium solution. It is stable, because for values of $y < 4.5$ the slope is positive, while for values $y > 4.5$ the slope is negative.

4. [12 points] The density of the Earth changes with the distance below the surface of the Earth one goes. If x gives the distance (in km) below the surface, the density $\delta(x)$ (in kg/km^3) is approximately

x	0	1000	2000	2900	3000	4000	5000	6000	6370
$\delta(x)$	3300	4500	5100	5600	10,100	11,400	12,600	13,000	13,000

(the radius R of the Earth is about 6370 km). Let r measure the distance out from the center of the Earth.

a. [4 points] The integral $\int_0^{4000} (4\pi \cdot r^2 \cdot \delta(R-r))\,dr$ is the limit as $\Delta r \to 0$ of a Riemann sum $\sum 4\pi \cdot r^2 \cdot \delta(R-r) \cdot \Delta r$. In the context of this problem, what do the terms of this sum represent?

Solution: The terms of the sum are "slices" of mass at a radius r with a thickness Δr. The expression $4\pi\, r^2\, \Delta r$ is the volume of a spherical shell with a radius r, and $\delta(R-r)$ is the density at that radius.

b. [4 points] Now consider the integral $\int_{R-4000}^{R} (4\pi \cdot r^2 \cdot \delta(R-r))\,dr$. Rewrite this in terms the variable x. Estimate your rewritten integral with MID(2).

Solution: We know r is the radial distance out from the center of the earth, so $x = R-r$, which gives $dx = -dr$, and we can substitute to rewrite the integral:

$$\int_{R-4000}^{R} 4\pi \cdot r^2 \cdot \delta(R-r)\,dr = -\int_{4000}^{0} 4\pi \cdot (R-x)^2 \cdot \delta(x)\,dx = \int_0^{4000} 4\pi \cdot (R-x)^2 \cdot \delta(x)\,dx.$$

Estimating this with MID(2), we have

$$\int_0^{4000} 4\pi \cdot (R-x)^2 \cdot \delta(x)\,dx \approx 4\pi \cdot (2000)\left((R-1000)^2\,(4500) + (R-3000)^2\,(10,100)\right)$$
$$\approx 6.144 \times 10^{15} \text{ kg}.$$

c. [4 points] Let $F(x) = \int_{R-x}^{R} (4\pi \cdot r^2 \cdot \delta(R-r))\,dr$. Find $F'(x)$, showing work that shows how you obtained your answer.

Solution: Using the Second Fundamental Theorem of Calculus,

$$F'(x) = -\frac{d}{dx}\int_R^{R-x} 4\pi\, r^2\, \delta(R-r)\,dr = -(-1)4\pi\,(R-x)^2\,\delta(x) = 4\pi\,(R-x)^2\,\delta(x).$$

5. [12 points] For each of the following series, carefully prove its convergence or divergence. You must clearly indicate what test(s) you use in your proof, and must carefully show all work that demonstrates their appropriateness and the calculations associated with the tests.

a. [6 points] $\displaystyle\sum_{n=1}^{\infty} \frac{2^n - 1}{e^n - n}$

Solution: There are a couple of possible methods we could use to show that this series converges. Using the ratio test, we look at $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. This is

$$\lim_{n\to\infty} \left|\frac{2^{n+1} - 1}{e^{n+1} - (n+1)} \cdot \frac{e^n - n}{2^n - 1}\right| = \lim_{n\to\infty} \left|\frac{e^n 2^n (2 - \frac{1}{2^n})(1 - \frac{n}{e^n})}{e^n 2^n (e - \frac{n+1}{e^n})(1 - \frac{1}{2^n})}\right| = \frac{2}{e}.$$

The limit is less than one, so by the ratio test we know that this series converges. Alternately, we could use the limit comparison test and compare with the geometric series $\sum \left(\frac{2}{e}\right)^n$; $\frac{2}{e} < 1$, so this is a convergent geometric series. Then we look at $\lim_{n\to\infty} \frac{b_n}{a_n}$, which is

$$\lim_{n\to\infty} \frac{2^n - 1}{e^n - n} \cdot \frac{e^n}{2^n} = \lim_{n\to\infty} \frac{2^n e^n (1 - \frac{1}{e^n})}{2^n e^n (1 - \frac{n}{e^n})} = 1.$$

This is finite, so the convergent properties of the two are the same, and thus our series converges.

b. [6 points] $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^3 + \cos(n)}$

Solution: Note that $a_n = \frac{n}{n^3+\cos(n)} \le \frac{n}{n^3-1} < \frac{n}{n^3 - \frac{1}{4}n^3} = \frac{4n}{3n^3} = \frac{4}{3n^2}$ for $n \ge 2$. Further $a_n > 0$. Thus, by comparison with the series $\frac{4}{3}\sum \frac{1}{n^2}$, which we know converges, we must have that this series converges.

Alternately, we could also use limit comparison with $\sum \frac{1}{n^2}$, which we know converges. The terms of the given and comparison series are positive, so we can use limit comparison. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^3 + \cos(n)} = \lim_{n\to\infty} \frac{1}{1 + \frac{\cos(n)}{n^3}} = 1.$$

Thus, by the limit comparison test we know that $\sum \frac{n}{n^3+\cos(n)}$ must converge.

6. [12 points] When a patient takes a drug (e.g., by ingesting a pill), the amount of the drug in her/his system changes with time. We can think of this process discretely (each pill is an immediately delivered dose) or continuously (each pill delivers a small amount of drug per unit time over a long time). This problem considers these two different models.

a. [4 points] Suppose that ibuprofen is taken in 200 mg doses every six hours, and that all 200 mg are delivered to the patient's body immediately when the pill is taken. After six hours, 12.5% of the ibuprofen remains. Find expressions for the amount of ibuprofen in the patient immediately before and after the nth pill taken. Include work; without work, you may receive no credit.

Solution: Immediately after the first pill, the amount in the patient's system is $A_1 = 200$. Immediately before the second pill is taken the amount in the patient's system is $B_2 = (0.125)(200)$, and immediately after, $A_2 = 200 + (0.125)(200)$. Similarly, we have $B_3 = (0.125)(200) + (0.125)^2(200)$ and $A_3 = 200 + (0.125)(200) + (0.125)^2(200)$, and so on. Thus immediately after the nth pill taken, the patient has

$$A_n = 200 + (0.125)(200) + \cdots + (0.125)^{n-1}(200) = \frac{200(1 - 0.125^n)}{1 - 0.125}\ \text{mg},$$

or about $228.57(1 - 0.125^n)$ mg of ibuprofen in her/his system, and immediately before the nth pill the patient has (for $n > 1$),

$$B_n = (0.125)(200) + \cdots + (0.125)^{n-1}(200) = \frac{(0.125)(200)(1 - 0.125^{n-1})}{1 - 0.125}\ \text{mg},$$

or about $28.571(1 - 0.125^{n-1})$ mg of ibuprofen in her/his system.

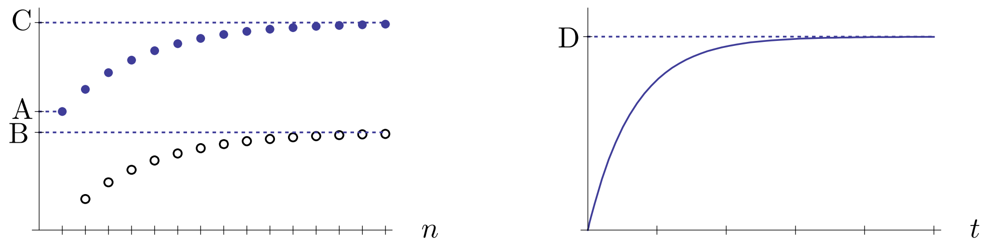
b. [4 points] Now suppose that ibuprofen is taken in a time-release capsule that continuously releases 35 mg/hr of ibuprofen per hour for six hours. The drug decays at a rate proportional to the amount in the body, with a constant of proportionality $r = 0.35$. Write a differential equation for the amount of ibuprofen, $y(t)$, in the patient as a function of time. Solve your differential equation, assuming that there is no ibuprofen in the patient initially.

Solution: We know that the rate of change of the ibuprofen in the body, $y(t)$, is 35 mg/hr less the decay of the amount that's present, so we have $\frac{dy}{dt} = 35 - 0.35y$, and if the initial amount present is zero, we also have $y(0) = 0$. We can solve this by separation of variables:

$$\frac{dy}{dt} = 35 - 0.35y = -0.35(y - 100), \quad \text{so} \quad \frac{dy}{y - 100} = -0.35\,dt.$$

Integrating both sides, we have $\ln|y - 100| = -0.35t + C$, so that $y = 100 + k\,e^{-0.35t}$. Because $y(0) = 0$, $k = -100$, and we have $y = 100(1-e^{-0.35t})$. Note that this is only valid for $0 \le t \le 6$; at $t = 6$ we expect that another pill will be taken, and we will then have to solve the same differential equation with the initial condition $y(6) = 100(1 - e^{-0.35(6)}) = 87.75$.

c. [4 points] *This is the third part of problem (6), which begins on the previous page.* The figure below shows graphs that may correspond to the solutions that you obtained in (6a) and (6b). The graph to the left corresponds to part (a), with the solid and open circles in the graph showing the amount of ibuprofen in the patient after and before the nth pill. The graph to the right corresponds to (b). Assuming that these graphs do represent your solutions, use your results from (a) and (b) to determine the values of A, B, C, and D.



Solution: We note that A is the amount of ibuprofen in the patient after taking one pill, which is 200 mg; C is the limiting value for this quantity, which is 228.6 mg. B is the limiting value for the amount of ibuprofen in the patient before the nth pill, which is 28.6 mg. D is the limit of the exponential in (b), which is 100 mg.

7. [16 points] The annual snowfall in Ann Arbor has mean 52.91 inches and standard deviation 12.67 inches. Assume that these data are normally distributed, and let x be the deviation of a year's snowfall from the mean (so that if $x = -2$ in a given year, it means that the snowfall that year was 50.91 inches). Then the probability density function for x is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/(2\sigma^2)} = 0.03149\, e^{-x^2/321.1},$$

so that its cumulative distribution function $P(x)$ is

$$P(x) = \int_{-\infty}^{x} p(t)\, dt = \int_{-\infty}^{x} 0.03149\, e^{-t^2/321.1}\, dt.$$

a. [2 points] Explain why $P(x) = \dfrac{1}{2} + \displaystyle\int_{0}^{x} 0.03149\, e^{-t^2/321.1}\, dt.$

Solution: We know the area under $p(x)$ is one, and because the normal distribution is symmetric about its mean, we know $\int_{-\infty}^{0} p(x)\, dx = \frac{1}{2}$. Thus $P(x) = \int_{-\infty}^{x} p(t)\, dt = \int_{-\infty}^{0} p(t)\, dt + \int_{0}^{x} p(t)\, dt = \frac{1}{2} + \int_{0}^{x} p(t)\, dt$.

This continues problem 7: here, $p(x) = 0.03149\, e^{-x^2/321.1}$, *and*

$$P(x) = \frac{1}{2} + \int_0^x 0.03149\, e^{-t^2/321.1}\, dt.$$

b. [5 points] Write a Taylor series for $p(x)$ (around $x = 0$).

Solution: We know that $e^x = 1 + x + \frac{1}{2!}\, x^2 + \cdots + \frac{1}{n!}\, x^n + \cdots$. Thus,

$$\begin{aligned} p(x) &= 0.03149\, e^{-x^2/321.1} \\ &= 0.03149 \left(1 - \frac{1}{321.1}\, x^2 + \frac{1}{2!\,(321.1^2)}\, x^4 + \cdots + \frac{(-1)^n}{n!\,(321.1^n)}\, x^{2n} + \cdots\right). \end{aligned}$$

c. [5 points] Write a Taylor series for $P(x)$ (around $x = 0$). *Hint: you will probably want to use your work from (b).*

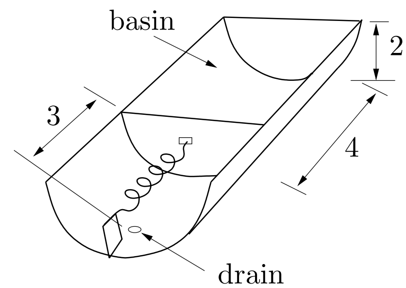
Solution: Using the result from (a) and integrating the series in (b), we have

$$\begin{aligned} P(x) &= \frac{1}{2} + \int_0^x 0.03149 \sum_{n=0}^{\infty} \frac{t^{2n}}{n!\, 321.1^n}\, dt \\ &= \frac{1}{2} + 0.03149 \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{n!\,(2n+1)\, 321.1^n}. \end{aligned}$$

d. [4 points] Use your result from (b) or (c) to estimate the probability that there will be up to 60 inches of snow this winter.

Solution: This probability is just $P(60 - 52.91) = P(7.09)$. Using the first two terms of the series in (c), we have $P(7.09) \approx \frac{1}{2} + 0.03149 \left(7.09 - \frac{(7.09)^3}{3\,(321.1)}\right) = 0.712$, or about a 71.2% chance. (The leading term gives 72.3%.)
We could also use the series for $p(x)$: $p(x) \approx 0.03149(1 - \frac{1}{321.1}\, x^2)$, so the probability we want is $\approx \frac{1}{2} + \int_0^{7.09} 0.03149(1 - \frac{1}{321.1}\, x^2)\, dx$, which will clearly be the same as the preceding. Obviously, trying to use $\int_{-\infty}^{7.09} 0.03149(1 - \frac{1}{321.1}\, x^2)\, dx$ does not work.

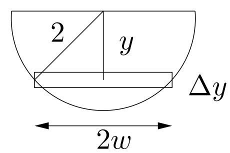
8. [12 points] Consider a semicircular basin with a movable end, as shown in the figure below. Suppose that the radius of the basin is 2 ft, and that its length when empty is 4 ft. The movable end to the basin is attached to a spring with a spring constant $k = 100$ lb/ft, and there is a drain hole 3 ft behind the resting position of the movable end.



a. [7 points] Suppose that we hold the movable end in the position shown in the figure and fill the basin with water (which weighs 62.4 lb/ft^3). What is the force of the water on the end of the basin we are holding?

Solution: Slicing the wall horizontally into slices of height Δy, where y measures the distance down from the top of the basin, we see that the width of the slice is $2w = 2\sqrt{4-y^2}$. Thus, noting that the pressure at that depth is $P = 62.4\,y$, the total force on the wall is found by integrating pressure times area from the top of the wall to the bottom, giving

$$F = \int_0^2 124.8\,y\,\sqrt{4-y^2}\,dy = -124.8\,(\tfrac{1}{3})\,(4-y^2)^{3/2}\Big|_0^2 = 41.6(8) = 332.8 \text{ lb.}$$



b. [5 points] If we do not hold the movable end of the basin and we add water at a rate of 4 ft^3/minute, and if water can leave through the drain hole at the same rate, how full will the basin get? Will it overflow?

Solution: The force required to push the spring back to the drain hole is $F = kx = 100(3) = 300$ lb, so the basin will fill only until the force exerted on the movable end is 300 lb. This is when

$$\begin{aligned} 300 &= \int_{y_0}^2 124.8\,(y-y_0)\,\sqrt{4-y^2}\,dy \\ &= -124.8\left((\tfrac{1}{3})\,(4-y^2)^{3/2}\Big|_{y_0}^2 - \int_{y_0}^2 y_0\,\sqrt{4-y^2}\,dy\right) \\ &= 124.8\left((\tfrac{1}{3})\,(4-{y_0}^2)^{3/2} - \int_{y_0}^2 y_0\,\sqrt{4-y^2}\,dy\right) \end{aligned}$$

We can evaluate the integral on the right with a trigonometric substitution, or we can approximate the result numerically; in either case we obtain $y_0 \approx 0.086$, so that the tank fills to a height of 1.914 ft.