Math 116 — First Midterm

October 14, 2009

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	10	
2	12	
3	12	
4	8	
5	12	
6	10	
7	11	
8	13	
9	12	
Total	100	

1. [10 points] Suppose that $f(x)$ is an odd function, $g(x)$ is an even function, and

- $\int_3^8 f(x)dx = 4$
- $\int_0^8 f(x)dx = 10$
- $\int_3^8 g(x)dx = -2$
- $\int_{-3}^3 g(x)dx = 5$

Determine each of the following quantities, if possible. If there is not enough information to determine the answer, then write "NI" in the space provided. You do not need to show your work for this page.

a. [2 points] Evaluate $\int_3^8 (f(x) - 3g(x))dx$.

10

b. [2 points] Evaluate $\int_{-8}^0 g(x)dx$.

0.5

c. [2 points] Evaluate $\int_3^8 f(x)g(x)dx$.

NI

d. [2 points] Evaluate $\int_{-2}^0 f(4x)dx$.

-2.5

e. [2 points] Evaluate $\int_0^3 (f(x) + 4)dx$.

18

2. [12 points] A population of creatures is placed on a small preservation space. Ten creatures are initially placed on the preservation. The time it takes for a population to reach C creatures is given by

$$T(C) = \int_{10}^{C} \frac{20dx}{x(400-x)},$$

where T is measured in years after the creatures were first placed on the preservation.

a. [6 points] Find a function for $T(C)$ by analytically solving the integral given above. Be sure to show all appropriate work.

Solution: First we use partial fractions to rewrite the integrand.

$$\frac{20}{x(400-x)} = \frac{A}{x} + \frac{B}{400-x} = \frac{400A - Ax + Bx}{x(400-x)}$$

This gives us the conditions $A = B = \frac{1}{20} = 0.05$. We then have

$$\begin{aligned} T(C) &= \frac{1}{20}\int_{10}^{C} \frac{dx}{x} + \frac{1}{20}\int_{10}^{C} \frac{dx}{400-x} \\ &= \frac{1}{20}\ln|x|\Big|_{10}^{C} - \frac{1}{20}\ln|400-x|\Big|_{10}^{C} \\ &= \frac{1}{20}\ln|C| - \frac{1}{20}\ln|10| - \frac{1}{20}\ln|400-C| + \frac{1}{20}\ln|390| \\ &= \frac{1}{20}\ln|39| + \frac{1}{20}\ln\left|\frac{C}{400-C}\right| \end{aligned}$$

b. [2 points] How long does it take for the creatures to reach a population of 50? State your answer in a complete sentence and include units in your answer.

Solution:

$$T(C) = \frac{1}{20}\ln|39| + \frac{1}{20}\ln\left|\frac{50}{350}\right| \approx 0.08588.$$

It takes approximately 0.08588 years (or approximately 1.0306 months) for the population of creatures to reach 50.

c. [4 points] Determine if the integral $T(400) = \int_{10}^{400} \frac{20dx}{x(400-x)}$ converges or diverges. What does your conclusion mean in terms of the creatures on the preservation?

Solution:

$$\begin{aligned} T(400) &= \frac{1}{20}\int_{10}^{400} \frac{dx}{x} + \lim_{b\to 400} \frac{1}{20}\int_{10}^{b} \frac{dx}{400-x} \\ &= \frac{1}{20}\ln|40| + \lim_{b\to 400}\left(-\frac{1}{20}\ln|400-b| + \frac{1}{20}\ln|390|\right) \end{aligned}$$

We know that $\lim_{b\to 400}\left(-\frac{1}{20}\ln|400-b|\right)$ diverges, so the integral diverges. This means that the time to reach 400 creatures is infinite, so the population will never reach 400 creatures.

3. [12 points] Let $g(x)$ and $h(x)$ be nonnegative functions defined for $x \geq 0$, let h be differentiable for $x \geq 0$ and suppose that

- $g(x) \leq 4$ when $x \geq 1$
- $\lim_{x\to\infty} h(x) = 0$
- $\lim_{x\to 0} \frac{h(x)}{x} = 0$
- $\int_0^\infty h(x)dx$ and $\int_0^\infty \frac{h(x)}{x^2}dx$ both converge
- $\int_0^\infty g(x)dx$ diverges

Indicate whether you think the following integrals converge, diverge, or whether there is not enough information to determine convergence. You do not need to show your work for this page.

a. [3 points] $\int_0^\infty g(x)h(x)dx$

CONVERGES DIVERGES $\boxed{\text{CANNOT TELL}}$

b. [3 points] $\int_1^\infty g(x)h(x)dx$

$\boxed{\text{CONVERGES}}$ DIVERGES CANNOT TELL

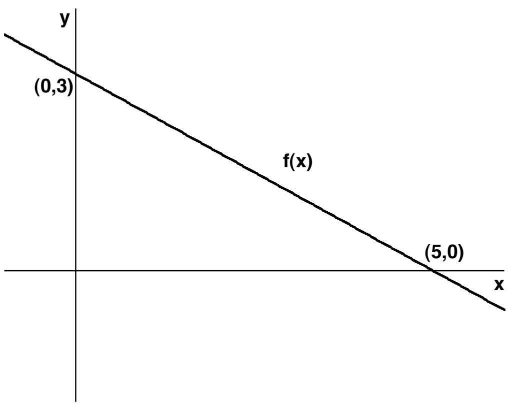
c. [3 points] $\int_0^\infty \frac{g(x)+h(x)}{2}dx$

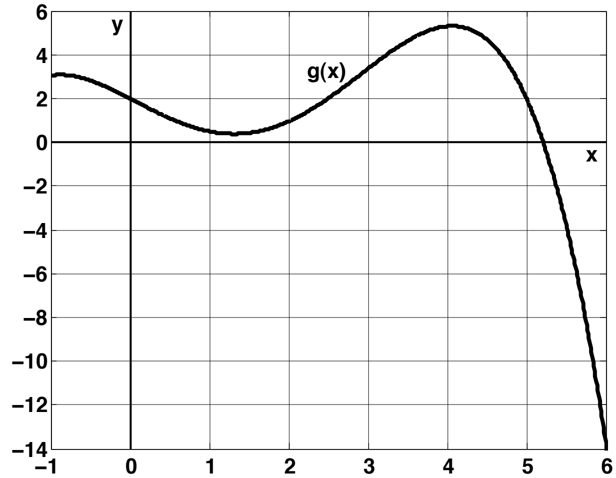
CONVERGES $\boxed{\text{DIVERGES}}$ CANNOT TELL

d. [3 points] $\int_0^1 \frac{h(x)}{x}dx$

$\boxed{\text{CONVERGES}}$ DIVERGES CANNOT TELL

4. [8 points] The graphs of $f(x)$ and $g(x)$ are shown below. Suppose that $f(x)$ is a linear function. Estimate $\int_0^5 f(x)g'(x)dx$. Be sure to show appropriate work to support how you derived your answer.





Solution: We can use integration by parts, letting $u = f(x), du = f'(x)dx, dv = g'(x)dx$, and $v = g(x)$. We then have

$$\int_0^5 f(x)g'(x)dx = f(x)g(x)|_0^5 - \int_0^5 f'(x)g(x)dx.$$

We know that $f(x)$ is a linear function, so from the graph we determine $f(x) = -\frac{3}{5}x + 3$ and $f'(x) = -\frac{3}{5}$. We also know $f(5) = 0, f(0) = 3, g(5) = 2$, and $g(0) = 2$. We can use this to solve in our expression above.

$$\int_0^5 f(x)g'(x)dx = f(5)g(5) - f(0)g(0) + \frac{3}{5}\int_0^5 g(x)dx = -6 + \frac{3}{5}\int_0^5 g(x)dx.$$

By counting boxes, we can approximate $\int_0^5 g(x)dx$, noting that each box has an area of 2. We approximate $\int_0^5 g(x)dx \approx 13$, and so we are left with

$$\int_0^5 f(x)g'(x)dx \approx -6 + \frac{3}{5}(13) = 1.8.$$

5. [12 points] A right isosceles triangle is a right triangle whose sides containing the right angle are of equal length. The length from the triangle's hypotenuse to its right-angle vertex (opposite of the hypotenuse) is half the length of the hypotenuse.
Consider the solid whose cross sections perpendicular to the x-axis are right isosceles triangles, where the hypotenuse of each triangular cross-section is contained in the region of the xy-plane bounded by the curves $y = \sin(x)$ and $y = -\sin(x)$ between $x = 0$ and $x = \pi$.

a. [3 points] Find the volume of the cross-sectional slice located at $x = x_i$ with thickness Δx.

Solution: The triangle has base length $2\sin(x_i)$ and the height length is $\sin(x_i)$. The area of the triangle is then $\sin^2(x_i)$, so the volume of the slice is $\sin^2(x_i)\Delta x$.

b. [3 points] Write a Riemann sum that approximates the volume of the entire solid using n cross-sectional slices.

Solution: We can approximate the area of the solid by finding the volume of n cross-sectional slices of depth Δx, and then adding these slices.

$$\text{volume} \approx \sum_{i=1}^{n} \sin^2(x_i)\Delta x$$

c. [6 points] Find the exact volume of the solid by using a definite integral.

Solution: As we let $n \to \infty$ in our Riemann sum, we approach the exact volume with a definite integral. We have volume $= \int_0^\pi \sin^2(x)dx$. We use integration by parts and some algebraic manipulation to solve the integral. Let $u = \sin(x), du = \cos(x)dx, dv = \sin(x)dx, v = -\cos(x)$. Then we have

$$\begin{aligned}
\int_0^\pi \sin^2(x)dx &= -\sin(x)\cos(x) + \int_0^\pi \cos^2(x)dx \\
&= -\sin(x)\cos(x)|_0^\pi + \int_0^\pi (1-\sin^2(x))dx \\
&= 0 + \int_0^\pi dx - \int_0^\pi \sin^2(x)dx \\
2\int_0^\pi \sin^2(x)dx &= \int_0^\pi dx \\
2\int_0^\pi \sin^2(x)dx &= x|_0^\pi \\
\int_0^\pi \sin^2(x)dx &= \frac{\pi}{2}
\end{aligned}$$

6. [10 points] The distance between two points (x_1, y_1) and (x_2, y_2) is given by

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Consider the curve described by $y = \sqrt{3x^2 - 3}$, over the domain $2 \le x \le 4$. What is the average distance of the points on this curve to the point $(2, 0)$?

Solution: A point on the curve has coordinates $(x, \sqrt{3x^2 - 3})$, so the distance from an arbitrary point on the curve to the point $(2, 0)$ is given by

$$\begin{aligned} D &= \sqrt{(x-2)^2 + (\sqrt{3x^2-3} - 0)^2} \\ &= \sqrt{x^2 - 4x + 4 + 3x^2 - 3} \\ &= \sqrt{4x^2 - 4x + 1} \\ &= \sqrt{(2x-1)^2} \\ &= 2x - 1 \end{aligned}$$

We can use a definite integral to find the average distance over the domain $2 \le x \le 4$.

$$\begin{aligned} \text{avg. distance} &= \frac{1}{4-2} \int_2^4 (2x-1)dx \\ &= \frac{1}{2}(x^2 - x)\Big|_2^4 \\ &= \frac{1}{2}(12 - 2) \\ &= 5 \end{aligned}$$

The average distance between a point on the curve $y = \sqrt{3x^2 - 3}$ over the domain $2 \le x \le 4$ is 5.

7. [11 points] A land surveyor is hired to measure the area of a plot of land to be sold. The surveyor uses two main highways as points of reference while measuring the property. Highway 116 is south of the property and runs perfectly in the east-west direction. Highway 1 is west of the property and runs perfectly in the north-south direction. The surveyor starts at Highway 1 and moves eastward for the entire four-mile width of the property as he measures the distances of the northern and southern borders of the property from Highway 116. Let $n(x)$ and $s(x)$ be the distances, in miles, of the northern border and southern borders, respectively, from Highway 116 when he is x miles east of Highway 1. The surveyor's measurements are recorded in the table below.

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$n(x)$	13.6	13.5	12.9	12.7	12.4	12.0	11.4	11.2	10.9
$s(x)$	7.8	8.1	8.2	8.5	8.7	8.8	9.1	10.0	10.9

a. [4 points] Estimate the area of the property using the midpoint rule with four subintervals. Be sure to show all appropriate work and don't forget to include appropriate units.

Solution: Using four subintervals, we have $\Delta x = 1$. To determine the length of each subinterval, we consider the distance $n(x) - s(x)$. If we use the midpoint rule to approximate the area, we then have

$$\begin{aligned} \text{MID}(4) &= 1[(n(0.5) - s(0.5)) + (n(1.5) - s(1.5)) + (n(2.5) - s(2.5)) + (n(3.5) - s(3.5))] \\ &= 1(5.4 + 4.2 + 3.2 + 1.2) = 14 \end{aligned}$$

Using the midpoint rule with four subintervals, we find that the area of the land is approximately 14 square miles.

b. [4 points] Estimate the area of the property using the trapezoid rule with four subintervals. Be sure to show all appropriate work and don't forget to include appropriate units.

Solution: Again, we have $\Delta x = 1$ and the lengths are determined by the distance $n(x) - s(x)$. The trapezoid rule uses the average of the left-hand and right-hand sums.

$$\text{RIGHT}(4) = 1[4.7 + 3.7 + 2.3 + 0] = 10.7 \qquad \text{LEFT}(4)1[5.8 + 4.7 + 3.7 + 2.3] = 16.5$$

$$\text{TRAP}(4) = \frac{10.7 + 16.5}{2} = 13.6$$

Using the trapezoid rule with four subintervals, we find that the area of the land is approximately 13.6 square miles.

c. [3 points] Because he took calculus, the surveyor knows that he can determine Δx, the uniform distance at which he should make measurements in order to ensure the measured area is within a desired level of accuracy. Given that $n(x)$ is a decreasing function and $s(x)$ is an increasing function, determine the value of Δx the surveyor should use in order to measure the area within 0.5 square miles if he is using left- and right-hand Riemann sum approximations for the area.

Solution: We solve for Δx under the condition $\Delta x \left|[n(4) - s(4)] - [n(0) - s(0)]\right| \leq 0.5$, which gives $\Delta x \leq 0.0862$ miles.

8. [13 points] Let $C(u)$ be a function that satisfies $C'(u) = \frac{\cos(u^2)}{u}$, $C(2) = 3$, and let $S(u)$ be a function that satisfies $S'(u) = \frac{\sin(u^2)}{u}$, $S(2) = -1$.

a. [4 points] Write expressions for $C(t)$ and $S(t)$ that satisfy the above conditions.

Solution:

$$C(t) = \int_2^t \frac{\cos(u^2)}{u} du + 3 \qquad S(t) = \int_2^t \frac{\sin(u^2)}{u} du - 1$$

b. [5 points] A particle traces out the curve given by the parametric equations $x(t) = C(\ln(t))$, $y(t) = S(\ln(t))$ for $t \geq 10$. What is the speed of the particle at time t? You may assume that $t \geq 10$.

Solution:

$$\begin{aligned} \text{speed} &= \sqrt{\left(\frac{d}{dt}\int_2^{\ln(t)} \frac{\cos(u^2)}{u} du\right)^2 + \left(\frac{d}{dt}\int_2^{\ln(t)} \frac{\sin(u^2)}{u} du\right)^2} \\ &= \sqrt{\left(\frac{1}{t}\cdot\frac{\cos(\ln(t))^2}{\ln(t)}\right)^2 + \left(\frac{1}{t}\cdot\frac{\sin(\ln(t))^2}{\ln(t)}\right)^2} \\ &= \frac{1}{t\ln(t)} \end{aligned}$$

c. [4 points] For $t \geq 10$, is the curve given by the parametric equations in part (b) of finite or infinite length? Justify your answer.

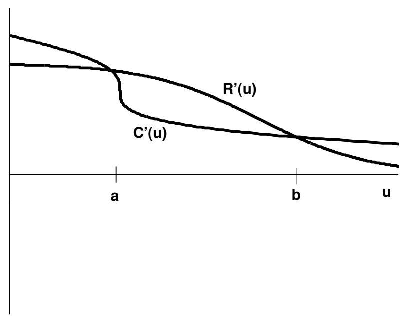
Solution: The curve has infinite length. The arc length for $t \geq 10$ is given by integrating the speed over the interval $[10, \infty)$. With a u-substitution $u = \ln(t)$ we have

$$\text{arc length} = \int_{10}^{\infty} \frac{dt}{t\ln(t)} = \int_{\ln(10)}^{\infty} \frac{du}{u}$$

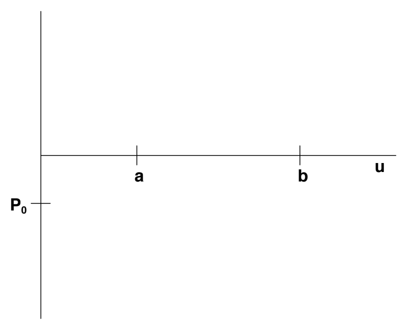
which diverges by the p-test.

9. [12 points] The primary objective of most manufacturing companies is to produce and sell the number of units that will generate the maximum profit for the company. Let $R(u)$ define the revenue income the company earns when selling u units, and let $C(u)$ define the cost of producing u units. Then the profit, P, of selling and producing u units is determined by $P(u) = R(u) - C(u)$, where profit, revenue, and cost are all measured in dollars.

a. [4 points] When trying to determine if it is beneficial to produce and sell additional goods, companies will often consider the marginal revenue, defined by $R'(u)$, and the marginal cost, defined by $C'(u)$. Below is a sketch of one company's marginal revenue and marginal cost, as a function u units. On the same axes, sketch a graph of the company's marginal profit, $P'(u)$.



b. [4 points] Using your answer to part (a), sketch a graph of $P(u)$ on the axes provided below, given the conditions that $P(0) = P_0$ and $P(b) > 0$.



c. [4 points] Given that $\int_a^b R'(u)du = \$135,000$, $\int_a^b C'(u)du = \$64,000$, and the company's profit when selling b units is \$52,000, determine the company's profit when selling a units. Does the company make or lose money when selling a units?

Solution:

$$\begin{aligned} P(b) - P(a) = \int_a^b P'(u)du &= \int_a^b R'(u)du - \int_a^b C'(u)du \\ 52,000 - P(a) &= 71,000 \\ P(a) &= -19,000 \end{aligned}$$

The company loses \$19,000 when selling a units.