Math 116 — Second Midterm

November 18, 2009

Name: EXAM SOLUTIONS

Instructor: ____________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	12	
2	12	
3	12	
4	14	
5	12	
6	9	
7	16	
8	13	
Total	100	

1. [12 points] A window is being rated for underwater pressure tests. The window surface is the region bounded by the x-axis, the line $x = 4$ and the graph of $y = x/2$, where all lengths are measured in feet. (Note that the density of water is 62.4 pounds per cubic foot.)

a. [4 points] Suppose the window is lowered under water so that the top of the window is F feet below the surface. Write an expression that estimates the force due to pressure on a thin horizontal strip of the window h_i feet below the top of the window, given that the width of the strip is Δh .

Solution: The force due to hydrostatic pressure is approximately given by the product of the density of water, the depth of the slice, and the area of the slice. A slice of width Δh that is h_i feet below the top of the window has depth $F + h_i$. The pressure along this strip is $62.4(F + h_i)$ pounds per square foot. The width of the strip of window h_i feet below the top point is given by $w_i = 2h_i$, and so the area of the strip is $2h_i\Delta h$. The force due to pressure along the strip is then given by

$$Force = Pressure \times Area = 62.4(F + h_i)(2h_i\Delta h) = 124.8h_i(F + h_i)\Delta h$$

b. [5 points] Set up a definite integral that represents the total force due to pressure on the window if the top of the window is F feet under water. Evaluate this integral in order to determine the total force due to pressure on the window. Show appropriate work to support your answer, and include units in your final answer. Your answer may be in terms of F.

Solution: The window is 2 feet deep at its deepest.

$$\begin{aligned} \text{TotalForce} &= \int_0^2 124.8h(F+h)dh \\ &= \int_0^2 (124.8hF + 124.8h^2)dh \\ &= (62.4Fh^2 + 41.6h^3)|_0^2 \\ &= 249.6F + 332.8 \end{aligned}$$

The total force on the window due to pressure is $249.6F + 332.8$ pounds.

c. [3 points] How deep must the top of the window be in order for there to be a total of 700 pounds of force exerted due to water pressure on the window?

Solution:

$$\begin{aligned} 700 &= 249.6F + 332.8 \\ 367.2 &= 249.6F \\ F &\approx 1.4712 \end{aligned}$$

The top of the window is approximately 1.47 feet below the surface of the water.

2. [12 points] On November 18, 2009, Gabrielle turned one year old. Starting today, Gabrielle's first birthday, and on every birthday up through her eighteenth birthday, Gabrielle's grandmother is going to deposit \$100 in an account that earns 6.7% interest, compounded annually. On the day she turns eighteen, her grandmother will make the last deposit of \$100, and then give Gabrielle all of the money in the account to help her pay for college.

a. [3 points] What is the present value of the deposit that will be made on Gabrielle's eighteenth birthday?

Solution: Gabrielle's eighteenth birthday is in 17 years, so the present value is

$$P = \frac{100}{(1.067)^{17}} \approx \$33.21.$$

b. [3 points] Write an expression for D_n, the present value of the deposit that will be made on Gabrielle's n^{th} birthday, for $1 \leq n \leq 18$.

Solution:

$$D_n = \frac{100}{(1.067)^{n-1}}$$

c. [3 points] Write an expression for B_n, the present value of the account's balance after the n^{th} deposit has been made.

Solution:

$$\begin{aligned} B_n &= 100 + \frac{100}{(1.067)^1} + \frac{100}{(1.067)^2} + ... + \frac{100}{(1.067)^{n-1}} \\ &= \sum_{i=1}^{n} \frac{100}{(1.067)^{i-1}} \end{aligned}$$

d. [3 points] Determine the present value of the account's balance after the final deposit is made on Gabrielle's eighteenth birthday.

Solution: This is a geometric series. The first term is $a = 100$ and the ratio of successive terms is $x = \frac{1}{1.067}$. There are 18 terms in the series, so we use

$$B_{18} = \frac{100\left(1 - \left(\frac{1}{1.067}\right)^{18}\right)}{1 - \frac{1}{1.067}} \approx 1096.94$$

The present value of the account in 18 years is approximately \$1096.94.

3. [12 points] Indicate whether you think each of the following series converges, diverges, or whether there is not enough information to determine convergence. You do not need to show your work for this page.

a. [3 points] Suppose $\sum_{n=1}^{\infty}(a_n + b_n)$ converges. Determine the convergence of $\sum_{n=1}^{\infty} a_n$.

CONVERGES DIVERGES [CANNOT TELL]

b. [3 points] Suppose $\sum_{n=1}^{\infty} a_n$ converges. Determine the convergence of $\sum_{n=1}^{\infty}(a_n + 4)$.

CONVERGES [DIVERGES] CANNOT TELL

c. [3 points] Suppose $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}$. Determine the convergence of $\sum_{n=1}^{\infty}\frac{a_n}{n}$.

[CONVERGES] DIVERGES CANNOT TELL

d. [3 points] Determine the convergence of $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$.

[CONVERGES] DIVERGES CANNOT TELL

4. [14 points] Using a rope, it takes 8 minutes to lift a 50-pound box of dirt from the ground to a height of 20 feet above the ground. As the box is lifted, dirt falls out of the box at a constant rate such that a total of 2 pounds of dirt has been lost when it reaches the final height of 20 feet.

a. [4 points] Let h be the height, in feet, of the box above the ground. Write an expression estimating the work done in raising the box from a height of h_i feet to a height of $h_i + \Delta h$ feet, ignoring the weight of the rope.

Solution: The change in weight will be $-\frac{1}{10}$ pound per foot lifted. The weight of the box after it has been lifted h_i feet off the ground is $50 - \frac{1}{10}h_i$. When the box is lifted from h_i feet to a height of $h_i + \Delta h$ feet, it is lifted Δh feet. The work done to lift the box this distance is then Work $\approx (50 - \frac{1}{10}h_i)\Delta h$ foot-pounds.

b. [3 points] Find the total amount of work done in raising the box of dirt from the ground to the final height of 20 feet, ignoring the weight of the rope.

Solution: The total work done is given by the definite integral

$$\text{Work} = \int_0^{20} (50 - \frac{1}{10}h_i)dh = 50h - \frac{1}{20}h^2|_0^{20} = 980$$

The total work in raising the box of dirt is 980 foot-pounds.

c. [7 points] Suppose the rope lifting the box of dirt weighs 1.5 pounds per foot and dangles from a platform that is 30 feet above the ground. Find the total work done to lift the box of dirt to a height of 20 feet above the ground, taking into account the weight of the rope.

Solution: Now we will find the work done to lift the rope. The bottom ten feet of rope (ten feet of rope closest to the box) will move the full 20 feet. The weight of this section of rope is $(1.5)(10) = 15$ pounds, so the work done on this part of the rope is $(15)(20) = 300$ foot-pounds.
The top 20 feet of the rope will not move the full 20 feet. Let y measure the distance from the top of the platform to a small slice of rope of size Δy. The weight of the small slice is $1.5\Delta y$ pounds. A piece that is y_i feet from the top moves y_i feet as the box is lifted. The work done to lift the small slice of rope is then $(1.5\Delta y)(y_i)$ foot-pounds. We can sum up all such slices, which leads to the definite integral

$$\text{Work} = \int_0^{20} 1.5y dy = 0.75y^2|_0^{20} = 300$$

So 300 foot-pounds of work are done to lift this part of the rope. The total work to lift the rope is 600 foot-pounds. The total work done to lift the box, including both the work done on the box and on the rope, is then 1580 foot-pounds.

5. [12 points] Consider the graph of the curve $r = f(\theta) = 2 + 2\cos(\theta)$.

a. [5 points] What is the minimal value of x taken by the points on the graph of $f(\theta)$? Show appropriate work to justify your answer.

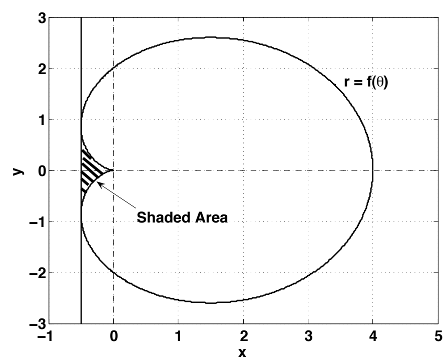
Solution: The x-coordinates corresponding to the point $(\theta, r(\theta))$ is $x(\theta) = r(\theta)\cos(\theta) = 2\cos(\theta) + 2\cos^2(\theta)$. To minimize, we take derivatives and find that

$$x'(\theta) = -2\sin(\theta) - 4\cos(\theta)\sin(\theta) = -2\sin(\theta)(1 + 2\cos(\theta)).$$

This last expression is zero when $\sin(\theta) = 0$, or $\theta = n\pi$ for all integers n, or $\cos(\theta) = -1/2$. By inspection of the graph, it is these zeroes that give the minimum value of x, corresponding to the angles $\theta = 2\pi/3$ and $4\pi/3$, with a minimum x-value of $-1/2$.

b. [3 points] Consider the vertical line $x = a$, where a is the value you found in part (a). Sketch the region bounded by $x = a$ and the graph of $f(\theta)$. Be sure to label your axes and graph appropriately.

Solution:



c. [4 points] Find inequalities for θ and r that describe the region bounded in part (b).

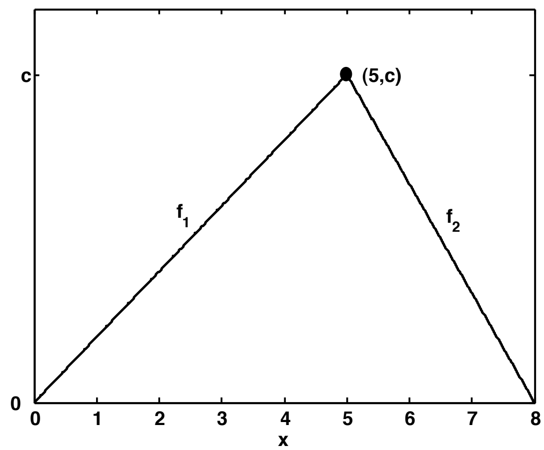
Solution: The region is bounded between the angles $\theta = 2\pi/3$ and $\theta = 4\pi/3$, the graph of $2 + 2\cos(\theta)$ and the line $x = -1/2$. We have to convert this last inequality into polar coordinates. Doing this, we have $r\cos(\theta) = -1/2$, or $r = -1/(2\cos(\theta))$. Therefore the inequalities describing the region are

$$\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}, \quad 2 + 2\cos(\theta) \le r \le -\frac{1}{2\cos(\theta)}$$

6. [9 points] Camp Summerama is a summer camp for teenagers. The camp is open for eight weeks every summer, and campers are able to attend for any length of time desired, between 0 to 8 weeks. The function $p(x)$ is the probability density function that the campers will enroll for x number of weeks. It is a piecewise function, defined by

$$p(x) = \begin{cases} \frac{c}{5}x & 0 \le x \le 5 \\ -\frac{c}{3}x + \frac{8c}{3} & 5 < x \le 8 \end{cases}$$

and shown in the graph below.



a. [2 points] What is the value of c?

Solution: $c = \frac{1}{4} = 0.25$

b. [3 points] Evaluate $p(7)$. Interpret your answer in a complete sentence, using the context of campers and weeks spent at camp.

Solution: Given that $c = \frac{1}{4}$, we have $p(7) = -\frac{1}{12}(7) + \frac{8}{12} = \frac{1}{12}$. For a small interval Δx, approximately $\frac{1}{12}\Delta x$ of the campers spent between 7 and $7 + \Delta x$ weeks at camp.

c. [4 points] Determine the median value for this density function. Interpret your answer in a complete sentence, using the context of campers and weeks spent at camp.

Solution:

$$0.5 = \int_0^T p(x)dx = \int_0^T \frac{1}{20}x dx = \frac{1}{40}x^2|_0^T = \frac{1}{40}T^2$$

$$T = \sqrt{20} \approx 4.472$$

Half of the campers spend less than 4.472 weeks at camp, and half spend more than 4.472 weeks at camp.

7. [16 points] Consider the following two functions defined by power series:

$$f(x) = \sum_{n=0}^{\infty} a_n(x+1)^n \text{ and } g(x) = \sum_{n=0}^{\infty} b_n(x-2)^n,$$

where $a_n, b_n > 0$ for every n, and $a_n \to 0$ as $n \to \infty$. Both $f(x)$ and $g(x)$ converge at $x = -2$, but $f(x)$ diverges at $x = 0$ and $g(x)$ diverges at $x = 7$.

a. [4 points] What is the interval of convergence for the function $f(x)$? Justify your answer using complete sentences.

Solution: The interval of convergence is $-2 \le x < 0$. Since $f(x)$ is based at $x = -1$, converges at $x = -2$, and diverges at $x = 0$, we know the radius of convergence is exactly one, and we know which endpoints are included in the interval.

b. [4 points] Give lower and upper bounds for the radius of convergence for the function $g(x)$. Justify your answer using complete sentences.

Solution: Similarly, $g(x)$ converges at $x = -2$ and is based at 2, so its radius of convergence is at least 4. However, since $g(x)$ diverges at $x = 7$, it has a radius of convergence of at most 5. So the radius of convergence of $g(x)$ is at least 4 and at most 5.

c. [4 points] Write the series $\sum_{n=0}^{\infty}\left(b_n + (-1)^n \frac{a_n}{2^n}\right)$ in terms of f and g to determine if this series converges, diverges, or it is impossible to tell. Give a brief explanation of your answer using complete sentences.

Solution: The series in question converges, since we can write the above series as

$$\sum_{n=0}^{\infty} b_n + (-1)^n \frac{a_n}{2^n} = \sum_{n=0}^{\infty} b_n(1)^n + a_n(-1/2)^n = g(3) + f(-3/2).$$

We know both $g(3)$ and $f(-3/2)$ converge because the inputs are within the radius of convergence for each function. Therefore the series converges.

d. [4 points] Suppose we also know that $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \frac{1}{5}$. What is the radius of convergence of the power series $h(x) = \sum_{n=0}^{\infty} b_{2n+1}x^n$? *(Hint: You might find it helpful to use the fact that* $\frac{a}{c} = \left(\frac{a}{b}\right)\left(\frac{b}{c}\right)$*.)*

Solution: Using the Ratio test, we have

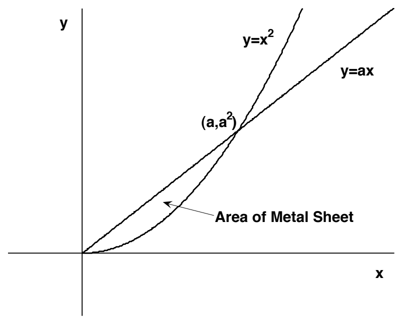
$$\begin{aligned}
\lim_{n\to\infty} |b_{2(n+1)+1}x^{n+1}/b_{2n+1}x^n| &= \lim_{n\to\infty} |b_{2n+3}x/b_{2n+1}||x| \\
&= \lim_{n\to\infty} |b_{2n+3}/b_{2n+2}| \cdot |b_{2n+2}/b_{2n+1}||x| \\
&= 1/25|x|
\end{aligned}$$

Therefore the radius of convergence is 25.

8. [13 points] Consider a solid metal sheet bounded by the curves $y = x^2$ and $y = ax$, for constant $a > 0$. The density of the sheet is given by $\delta(x) = 4$ grams per square centimeter.

a. [3 points] Sketch the area of the metal sheet in the space provided below. Be sure to label your graphs and axes.

Solution:



b. [4 points] Find the exact mass of the sheet, and be sure to include appropriate units. Your answer may be in terms of a.

Solution: Since the density is constant throughout the region, mass is the product of the area of the sheet and its density.

$$\text{Mass} = 4\int_0^a (ax - x^2)dx = 4\left(\frac{a}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^a = 2a^3 - \frac{4a^3}{3} = \frac{2a^3}{3}$$

The mass of the sheet is $\frac{2a^3}{3}$ grams.

c. [3 points] Find $\bar{x}$, the x-coordinate for the center of mass. Your answer may be in terms of a.

Solution:

$$\text{Moment} = \int_0^a 4x(ax - x^2)dx = \int_0^a (4ax^2 - 4x^3)dx = \left(\frac{4ax^3}{3} - x^4\right)\Big|_0^a = \frac{4}{3}a^4 - a^4 = \frac{1}{3}a^4$$

So $\bar{x} = \frac{\frac{a^4}{3}}{\frac{2a^3}{3}} = \frac{1}{2}a$.

d. [3 points] Find $\bar{y}$, the y-coordinate for the center of mass. Your answer may be in terms of a.

Solution:

$$\text{Moment} = \int_0^{a^2} 4y(\sqrt{y} - \frac{y}{a})dy = \int_0^{a^2}\left(4y^{3/2} - \frac{4}{a}y^2\right)dy = \left(\frac{8}{5}y^{5/2} - \frac{4}{3a}y^3\right)\Big|_0^{a^2} = \frac{4}{15}a^5$$

So $\bar{y} = \frac{\frac{4}{15}a^5}{\frac{2}{3}a^3} = \frac{2}{5}a^2$.