

Math 116 — Final Exam

December 17, 2009

Name: _____ EXAM SOLUTIONS _____

Instructor: _____ Section: _____

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	10	
2	8	
3	6	
4	12	
5	14	
6	12	
7	14	
8	12	
9	12	
Total	100	

You may find the following expressions useful. And you may not. But you may use them if they prove useful.

“Known” Taylor series (all around $x = 0$):

$$\begin{aligned}\sin(x) &= x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \\ \cos(x) &= 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \\ e^x &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \\ (1+x)^p &= 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots\end{aligned}$$

“Known” integral expressions:

$$\begin{aligned}\int x^n \ln x \, dx &= \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C \\ \int e^{ax} \sin(bx) \, dx &= \frac{1}{a^2 + b^2} e^{ax} (a \sin(bx) - b \cos(bx)) + C \\ \int e^{ax} \cos(bx) \, dx &= \frac{1}{a^2 + b^2} e^{ax} (a \cos(bx) + b \sin(bx)) + C \\ \int \sin(ax) \sin(bx) \, dx &= \frac{1}{b^2 - a^2} (a \cos(ax) \sin(bx) - b \sin(ax) \cos(bx)) + C, \quad a \neq b \\ \int \cos(ax) \cos(bx) \, dx &= \frac{1}{b^2 - a^2} (b \cos(ax) \sin(bx) - a \sin(ax) \cos(bx)) + C, \quad a \neq b \\ \int \sin(ax) \cos(bx) \, dx &= \frac{1}{b^2 - a^2} (b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx)) + C, \quad a \neq b\end{aligned}$$

“Known” equations from geometry:

$$\begin{aligned}\text{Volume of a sphere: } &V = \frac{4}{3} \pi r^3 \\ \text{Surface area of a sphere: } &A = 4\pi r^2 \\ \text{Volume of a cylinder: } &V = \pi r^2 h \\ \text{Volume of a cone: } &V = \frac{1}{3} \pi r^2 h\end{aligned}$$

1. [10 points] For each statement below, circle TRUE if the statement is *always* true; otherwise, circle FALSE. No partial credit on this page.

a. [2 points] The differential equation $\frac{dy}{dt} = y \sin(t + 1) - y$ is separable.

 True False

b. [2 points] If money is placed into a bank account with continuous interest rate k , then the amount of money, A , at time t years can be modeled with the differential equation $\frac{dA}{dt} = kt$.

 True False

c. [2 points] Suppose the power series $\sum_{n=1}^{\infty} C_n(x + 2)^n$ converges at $x = -5$, but diverges at $x = 5$. Then the series must diverge at $x = 3$.

 True False

d. [2 points] The differential equation $\frac{dy}{dx} = \cos(y)$ has an infinite number of equilibrium solutions.

 True False

e. [2 points] Consider the differential equation $\frac{dy}{dx} = x^2$, and the solution that satisfies $y(-1) = 1$. If Euler's method is used with step-size $\Delta x = 0.1$, then the Euler approximation for $y(-0.5)$ is an underestimate of the real solution.

 True False

2. [8 points] Eight differential equations are listed below in **A-H**. The slope fields of four of these eight differential equations are shown in the figures below. For each figure, write the letter (A-H) of the corresponding differential equation in the space provided below the figure. You do not need to show your work for this problem.

A. $y' = x^2 + y^2$

B. $y' = e^{y^2}$

C. $y' = x^2 - y^2$

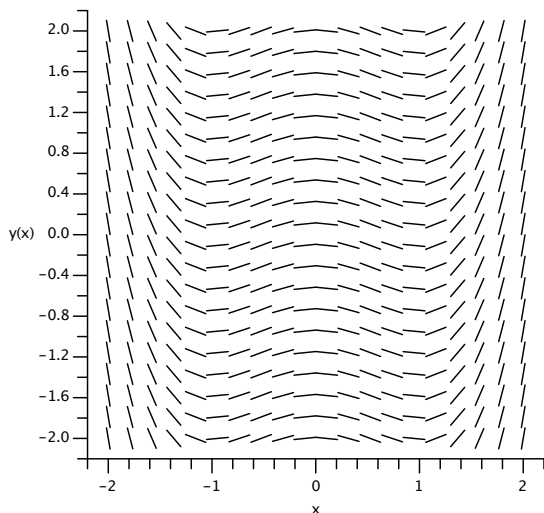
D. $y' = \cos(y)$

E. $y' = x^2 - 1$

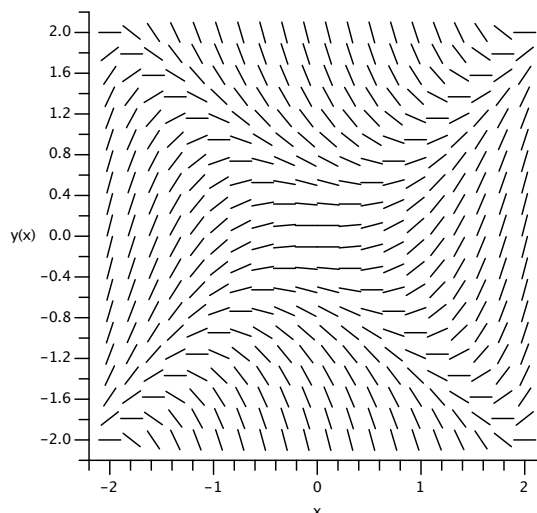
F. $y' = e^y$

G. $y' = x^3 - x$

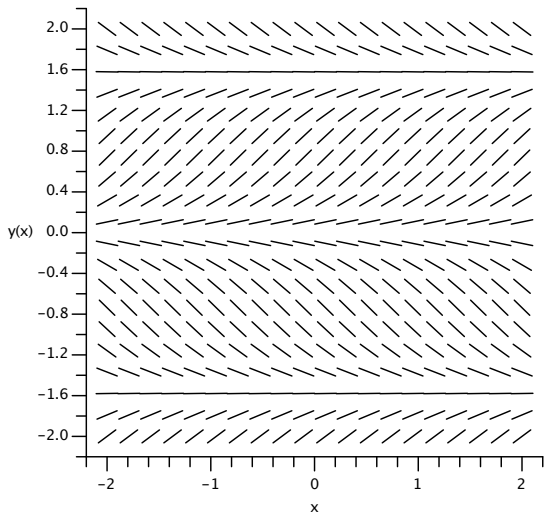
H. $y' = \sin(2y)$



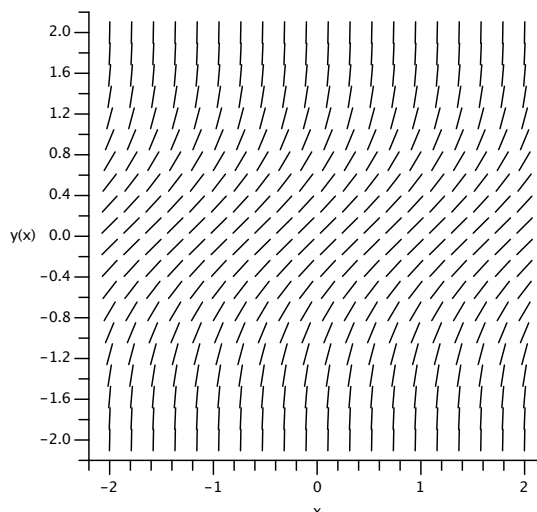
Differential Equation: _____ **G**



Differential Equation: _____ **C**



Differential Equation: _____ **H**



Differential Equation: _____ **B**

3. [6 points] Suppose $f(x)$ is a twice-differentiable function. On the interval $[a, b]$, for $0 < a < b$, $f(x)$ is positive, increasing, and concave up. Suppose $g(x) = x(f(x))^2$. If one uses the midpoint rule to estimate $\int_a^b g(x)dx$, will the estimation be an overestimate or an underestimate? Be sure to justify your answer and show all appropriate work. (*Hint: You might find it helpful to consider the concavity of function $g(x)$.*)

Solution: The midpoint rule is an overestimate when the integrand function is concave down, and an underestimate when the integrand function is concave up. In order to determine the concavity of $g(x)$ on the interval $[a, b]$, we need to find $g''(x)$.

$$\begin{aligned}g'(x) &= 2xf(x) \cdot f'(x) + (f(x))^2 \\g''(x) &= 2x(f(x)f''(x) + (f'(x))^2) + 2f(x)f'(x) + 2f(x) \cdot f'(x) \\&= 2xf(x)f''(x) + 2x(f'(x))^2 + 4f(x)f'(x)\end{aligned}$$

Since all terms are positive on the interval $[a, b]$, $g(x)$ is concave up, so the estimation will be an underestimate.

4. [12 points] During a party, the host discovers that he has been robbed of his favorite gold hallway clock, and immediately calls the police. The last time the host noticed the clock was still hanging, it was 6:00 p.m. When the police arrive, Officer Tom notices a decorative ice sculpture. When he first arrives at 9:00 p.m., the ice sculpture's temperature is 27°F. After questioning others at the party, Officer Tom again takes the temperature of the ice sculpture at 10:00 p.m., and finds it to be 29°F. The temperature of the room has remained a constant 68°F all day.

- a. [2 points] Let S be the temperature of the sculpture, measured in degrees Fahrenheit. Assuming the sculpture obeys Newton's Law of Heating and Cooling, write a differential for $\frac{dS}{dt}$, where t is the number of hours since 9:00 p.m. Your answer may contain an unknown constant, k .

Solution: $\frac{dS}{dt} = k(S - 68)$ (or some appropriate variation thereof)

- b. [7 points] Using separation of variables, and the information provided about the sculpture, solve the differential equation to find $S(t)$, where t is the number of hours since 9:00 p.m. Your answer should contain no unknown constants.

Solution: $\frac{dS}{S-68} = kdt$, which gives $\ln|S - 68| = kt + C$, or $S = Ae^{kt} + 68$. When $t = 0$, $S = 27$, which gives us $27 = A + 68$, so $A = -41$, leaving $S = -41e^{kt} + 68$. We use the other condition to solve for k . At $t = 1$, $S = 29$, so $29 = -41e^{k(1)} + 68$. We solve for k : $41e^k = 39$, or $e^k = \frac{39}{41}$, which leaves us with $k = \ln\left(\frac{39}{41}\right) \approx -0.05001$.

$$S = -41e^{-0.05001t} + 68$$

- c. [3 points] Company FunIce provided the sculpture, and it was delivered by their employee Bill. For each ice sculpture they produce, the company guarantees the temperature will be exactly 18°F upon delivery. If the sculpture was 18°F at delivery, and Bill left directly afterwards, should he be considered as a possible suspect of the robbery? Briefly justify your answer.

Solution: We solve for when the temperature of the sculpture was $S = 18$. We solve for t : $18 = -41e^{-0.05001t} + 68$, or $\frac{50}{41} = e^{-0.05001t}$. This gives us $t \approx -3.96823$ hours. The sculpture was delivered nearly four hours before 9:00 p.m., putting the delivery near 5:00 p.m. Since Bill left immediately after delivery, he should not be considered as a suspect, since the clock was still there at 6:00 p.m.

5. [14 points] A water filtration system has a rectangular basin 0.2 m wide, 0.4 m long and 0.2 m deep in which unfiltered water is poured. Along the bottom of the basin there is a filter, in the shape of a square, measuring 0.01 m on each side.

- a. [3 points] The density of water is 1000 kg/m^3 , and the gravitational constant is 9.8 m/sec^2 . Suppose the depth of the water in the basin is h m. What is the force due to water pressure on the filter? Include units in your answer.

Solution: Since the filter is on the bottom of the basin, at constant depth, the pressure is $P = \text{density} \cdot \text{gravity} \cdot \text{depth} = (1000 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(hm) = 9800 \text{ kg/(m} \cdot \text{s}^2) = 9800 \text{ Pa}$. The force is the pressure on the total area, so $F = P \cdot \text{area} = (9800)(.0001) = 0.98 \text{ N}$.

- b. [3 points] Write an equation for the volume of water in the basin, V , when the water is h m deep. Then use this equation to write an equation for the rate of change of volume of water, $\frac{dV}{dt}$, in terms of the rate of change of depth of water, $\frac{dh}{dt}$.

Solution: $V = (0.2)(0.4)h = 0.08h$, so $\frac{dV}{dt} = 0.08 \frac{dh}{dt}$

- c. [5 points] The rate at which water passes out of the basin is proportional to the square of the force due to water pressure exerted on the filter, with constant of proportionality $k > 0$. Suppose the basin is filled with water at a constant rate $r \text{ m}^3/\text{sec}$. Write a differential equation for the depth h of the water in the basin. That is, find an equation for $\frac{dh}{dt}$ in terms of h and constants r and k .

Solution: Letting V be the volume of water in the tank, we have

$$\text{rate of change of volume} = \text{rate in} - \text{rate out.}$$

The rate of change into the basin is r , and the rate of flow out is given to be proportional to the square of the force due to pressure in the basin found in part (a) with proportionality constant k . Therefore, the differential equation for the volume is $\frac{dV}{dt} = r - k(0.98h)^2$. Since $\frac{dV}{dt} = 0.08 \frac{dh}{dt}$, the differential equation for h is therefore $0.08 \frac{dh}{dt} = r - 0.9604kh^2$, so

$$\frac{dh}{dt} = 12.5r - 12.005kh^2$$

- d. [3 points] Suppose when the basin is filled at a constant rate of $0.0002 \text{ m}^3/\text{sec}$, the depth remains constant at 0.1 m. What is the constant of proportionality k ?

Solution: This is saying that the equation above has an equilibrium solution $h = 0.1$ when $r = 0.0002$. Using the differential equation, this means that $0 = 12.5(0.0002) - 12.005k(0.1)^2$, so $k \approx 0.020825$.

6. [12 points] A certain sculpture is the volume of revolution of the function $y = f(x)$ about the x -axis. The function $f(x)$ is twice differentiable, increasing, concave-up, and has the values given below, where all measurements are in meters:

x	0	0.25	0.5	0.75	1
$f(x)$	0.2	0.25	0.3	0.4	0.6

- a. [3 points] Suppose the density of the sculpture is 2400 kg/m^3 . Write an integral expression for the total mass of the sculpture.

Solution: $\int_0^1 2400\pi f(x)^2 dx$

- b. [3 points] Approximate the total mass of the sculpture using the *midpoint* rule with as many subdivisions as possible, given the data. Include appropriate units.

Solution: The most accurate use of the midpoint rule has only two subintervals. So

$$\begin{aligned} \int_0^1 2400\pi f(x)^2 dx &\approx 2400\pi \cdot 0.5 \cdot (f(0.25)^2 + f(0.75)^2) \\ &= 2400\pi \cdot 0.5 \cdot (0.625 + 0.16) \\ &\approx 838.805 \text{ kg} \end{aligned}$$

- c. [3 points] Approximate the moment of the sculpture about its axis of symmetry using the *trapezoid* rule with as many subdivisions as possible, given the data.

Solution:

$$\begin{aligned} \int_0^1 2400\pi x f(x)^2 dx &\approx 2400\pi \cdot 0.25 \cdot \left(\frac{1}{2} 0 f(0)^2 + 0.25 f(0.25)^2 + 0.5 f(0.5)^2 \right. \\ &\quad \left. + 0.75 f(0.75)^2 + \frac{1}{2} 1 f(1)^2 \right) \\ &\approx 679.762 \end{aligned}$$

- d. [3 points] Once completed, the sculpture will be arranged so that the end corresponding to $x = 0$ will rest on the ground. If the center of mass is more than half of the height of the sculpture, then the sculptor will have to brace the sculpture with extra supports. Approximate how high (in meters) the center of mass will be above the ground once the sculpture is placed on its $x = 0$ end using your answers from the previous two parts. Will the sculptor need to provide extra support?

Solution:

$$\bar{x} = 679.762/838.805 \approx 0.8104$$

The center of mass is approximately 0.8104 meters above the ground, which is more than 0.5 meters, half the height of the sculpture. The sculptor will need to provide extra support.

7. [14 points] Consider the function $g(x) = \int_0^{x^2} e^{-t^2} dt$.

a. [4 points] Suppose $f(x) = g'(x)$. Find a formula for $f(x)$.

Solution:

$$f(x) = e^{-(x^2)^2} \cdot (2x) = 2xe^{-x^4}$$

b. [6 points] Find the Taylor series of $f(x)$ about $x = 0$. Write your answer using summation (sigma) notation, including proper indices.

Solution:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ e^{-x^4} &= \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!} \\ f(x) &= 2x \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!} \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2x^{4n+1}}{n!} \end{aligned}$$

c. [4 points] Find the Taylor series of $g(x)$ about $x = 0$. Write your answer using summation (sigma) notation, including proper indices.

Solution: Starting with the Taylor series for e^x we have

$$\begin{aligned} g(x) &= \int g'(x) dx \\ &= \int f(x) dx = \sum_{n=0}^{\infty} (-1)^n \frac{2x^{4n+2}}{(4n+2)n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)n!} \end{aligned}$$

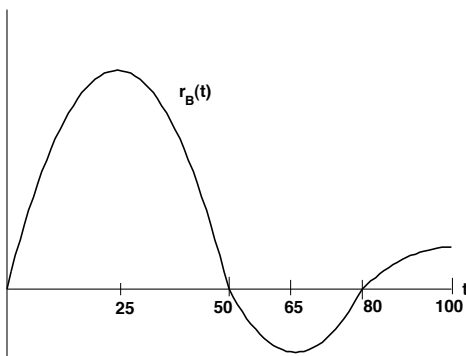
8. [12 points] Suppose the functions $r_A(t)$, $r_B(t)$, and $r_C(t)$ give the rate of population growth in millions of people per year t years after 1900 for Country A, Country B, and Country C, respectively.

- a. [4 points] In 1900, Country A's population was 2.3 million people. Write an expression for the function $P_A(Y)$, Country A's population in millions of people Y years after 1900.

Solution:

$$P_A(Y) = \int_0^Y r_A(t) dt + 2.3$$

- b. [4 points] Below is a graph of the rate of population growth for Country B, $r_B(t)$. Estimate the year in which Country B's population reached its greatest level and the year in which it reached its lowest level.



Year of greatest population: 1950 Year of lowest population: 1900

- c. [4 points] Suppose $r_C(t) = te^{-0.05t^2}$. During the years 1900-2000, what is the average growth rate for Country C? You must show enough work to justify your answer and include units to receive full credit on this problem. Write your final answer in the space provided below.

Solution: Average rate of growth = $\frac{1}{100} \int_0^{100} r_C(t) dt = \frac{1}{100} \int_0^{100} te^{-0.05t^2} dt = 0.1$.

Average growth rate: 0.1 millions people per year

9. [12 points] Consider the following series, all of which converge:

A. $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ B. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n+1}$ C. $\sum_{n=1}^{\infty} \frac{n+1}{n^3}$

D. $\sum_{n=1}^{\infty} \frac{2(-1)^n}{n}$ E. $\sum_{n=1}^{\infty} \frac{1+e^{-n}}{n!}$ F. $\sum_{n=1}^{\infty} \frac{10+\cos(n)}{n^2}$

Answer the following questions considering these series. For each question, you should list **ALL** answers that are possible on the space provided. List only the corresponding letter to the series you wish to use in your answer. For instance, if you wanted to include the series $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ in your answer, you should only write the letter “A”. You do not need to show your work for this problem.

a. [3 points] For which of the above series is it appropriate to use the Limit Comparison Test if comparing to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$? Write all possible answers on the line provided below. If no series satisfies this condition, write “none”.

_____ **A, C**

b. [3 points] For which of the above series is it appropriate to use the Ratio Test? Write all possible answers on the line provided below. If no series satisfies this condition, write “none”.

_____ **B, E**

c. [3 points] For which of the above series is it appropriate to use the Alternating Series Test? Write all possible answers on the line provided below. If no series satisfies this condition, write “none”.

_____ **B, D**

d. [3 points] Which of the above series is conditionally convergent? Write all possible answers on the line provided below. If no series satisfies this condition, write “none”.

_____ **D**