

# Math 116 — Final Exam

December 17, 2010

Name: \_\_\_\_\_

Instructor: \_\_\_\_\_ Section: \_\_\_\_\_

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a  $3'' \times 5''$  note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

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Problem	Points	Score
1	11	
2	7	
3	11	
4	11	
5	9	
6	8	
7	14	
8	14	
9	15	
Total	100	

You may find the following expressions useful. And you may not. But you may use them if they prove useful.

**“Known” Taylor series (all around  $x = 0$ ):**

$$\sin(x) = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots$$

1. [11 points] Indicate if each of the following is true or false by circling the correct answer. Justify your answer.

a. [2 points] If the radius of convergence of the power series  $\sum_{n=0}^{\infty} a_n(x-4)^n$  is 2, then  $\sum_{n=0}^{\infty} a_n$  diverges.

True                  False

b. [2 points] If  $P(x)$  is a cumulative distribution function with  $P(0) = \frac{1}{3}$ , then the median is positive.

True                  False

c. [3 points]

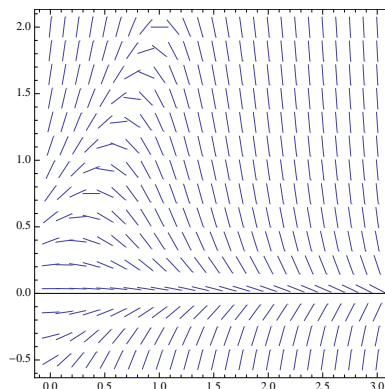
If  $F(x) = \int_{-x^2}^0 \frac{1}{1+t^4} dt$  then  $F(x)$  is decreasing for  $x > 0$ .

True                  False

d. [2 points] The differential equation  $y' = (y - x^3)y$  has two equilibrium solutions,  $y = 0$  and  $y = x^3$ .

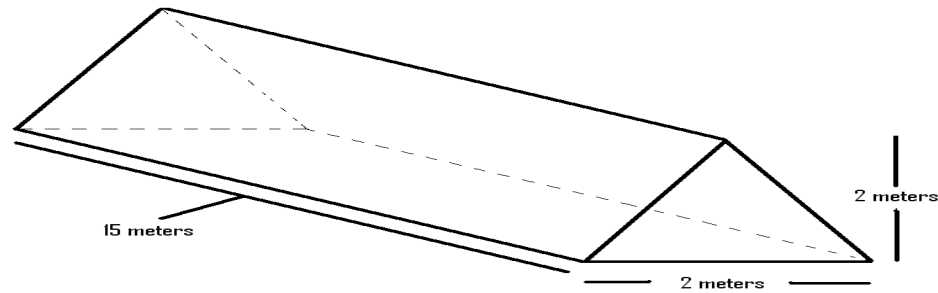
True                  False

e. [2 points] Using the slope field below, we can guarantee that the solution with initial condition  $y(0) = \frac{1}{2}$  satisfies  $y(3) < 0$ .



True                  False

2. [7 points] Deep beneath Dennison Hall lies a large septic tank. It has the shape of a triangular prism with the dimensions depicted below.

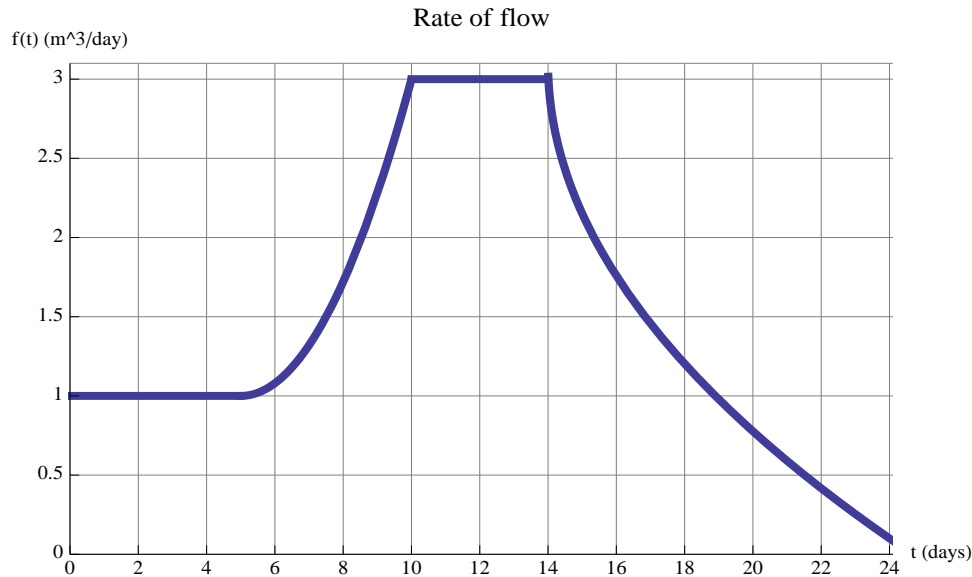


Suppose that the tank described above is full of sewage and that this sewage has a density of  $1000(1 + e^{-2x}) \frac{\text{kg}}{\text{m}^3}$ , where  $x$  is the distance in meters above the base of the tank.

- a. [5 points] Find a definite integral that computes the mass of the sewage in the tank.

- b. [2 points] Compute the value of the integral using your calculator. Do not forget to include the units.

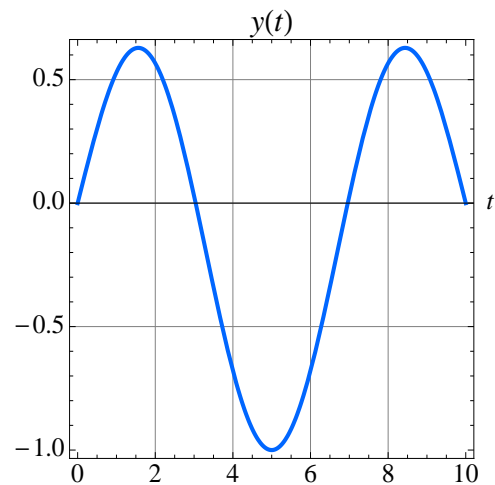
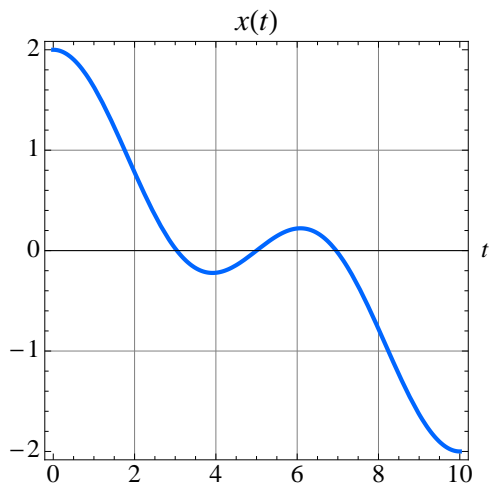
3. [11 points] Sewage flows into the tank described in the previous problem at a rate of  $f(t)$  cubic meters per day. Let  $t$  be the number of days since December 1, when the tank had  $1 m^3$  of sewage. A graph of  $f(t)$  is given below. Use it to answer the following questions.



- a. [3 points] Suppose that  $V(t)$  gives the volume of sewage in the tank at time  $t$ . Find a formula for  $V(t)$  in terms of  $f(t)$ .
- b. [2 points] For what times  $t$  in  $[0, 24]$  is  $V(t)$  concave up? \_\_\_\_\_
- c. [2 points] For what times  $t$  in  $[0, 24]$  is  $V(t)$  concave down? \_\_\_\_\_
- d. [4 points] Fill out the table below. Using the values in your table, compute Riemann sums with 3 subintervals to find an underestimate and an overestimate for  $V(12)$ . Justify why the Riemann sums you selected yield the appropriate under and upper estimates. Do not forget to include the units in your answer.

$t$	0	4	8	12
$f(t)$				

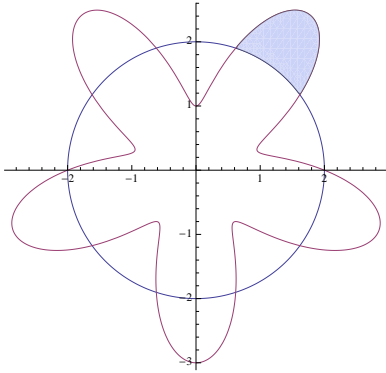
4. [11 points] A particle is moving in the  $x$ - $y$  plane according to the parametric equations  $(x(t), y(t))$  for  $0 \leq t \leq 10$ . The graph of these functions are shown below.



- a. [2 points] What are the starting and ending points of the particle?
- b. [3 points] At which values of  $0 < t < 10$  is the particle moving horizontally straight to the right or to the left?
- c. [2 points] At which values of  $0 < t < 10$  is the particle moving straight up or down?
- d. [2 points] At which values of  $t$  is the particle at the origin?
- e. [2 points] If  $v(t)$  is the speed of the particle at time  $t$ , which one is larger  $v(2)$  or  $v(5)$ ? Explain.

5. [9 points]

- a. [3 points] In the picture below, the graphs of  $r = 2$  and  $r = 2 - \sin(5\theta)$  are shown. Write a definite integral that computes the shaded area.



- b. [6 points] Find parametric equations of the tangent line to the limaçon  $r = \sin(\theta) - \frac{1}{2}$  at  $\theta = \frac{\pi}{4}$ .

6. [8 points] Let  $P(t)$  be the population of birds living in a lake  $t$  days after January 1, 2009. It has been noticed that the rate of growth of the population of birds varies depending on the season of the year. To take this into consideration, we assume the rate of growth of the population is equal to  $k(t)P$ , where  $k(t) = \frac{1}{100} \sin\left(\frac{2\pi}{365}t\right)$ . Hence  $P(t)$  satisfies

$$\frac{dP}{dt} = k(t)P$$

There were 100 birds living in the lake in January 1, 2009.

- a. [6 points] Solve the differential equation satisfied by  $P(t)$  and find the population of birds after 100 days after January 1, 2009.

- b. [2 points] Graph the solution  $P(t)$  in your calculator and use this graph to answer the following questions:
1. What is the maximum and minimum amount of birds living in the lake throughout the year?
  2. When are the maximum and minimum expected to occur?



7. [14 points] For each of the following sequences

1. Compute  $\lim_{n \rightarrow \infty} a_n$ .

2. Decide if  $\sum_{n=0}^{\infty} a_n$  converges or diverges. Circle your answer.

Support your answer by stating the test(s) or facts you used to prove convergence or divergence, and show complete work and justification.

a. [4 points]

$$a_n = \left(\frac{-1}{\pi}\right)^n \quad \lim_{n \rightarrow \infty} a_n = \text{_____} \quad \sum_{n=0}^{\infty} a_n : \quad \text{Converges} \quad \text{Diverges}$$

b. [4 points]

$$a_n = \frac{n^2 + 2}{1 + 4n^2} \quad \lim_{n \rightarrow \infty} a_n = \text{_____} \quad \sum_{n=0}^{\infty} a_n : \quad \text{Converges} \quad \text{Diverges}$$

c. [6 points]

$$a_n = \frac{n}{\sqrt{n^4 + 5}} \quad \lim_{n \rightarrow \infty} a_n = \text{_____} \quad \sum_{n=0}^{\infty} a_n : \quad \text{Converges} \quad \text{Diverges}$$

8. [14 points] Consider the following power series

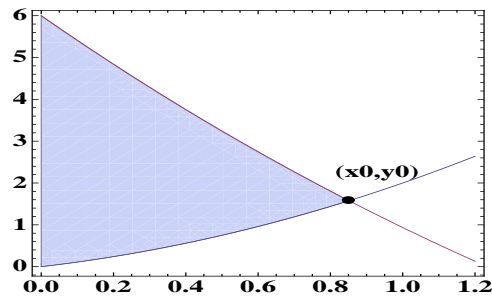
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n(2n+1)}(x-3)^n$$

a. [11 points] For what values of  $x$  does the power series converge?

b. [2 points] For what values of  $x$  does the power series converges absolutely?

c. [1 point] For what values of  $x$  does the power series converges conditionally?

9. [15 points] The graph shows the area between the graphs of  $f(x) = 6 \cos(\sqrt{2x})$  and  $g(x) = x^2 + x$ . Let  $(x_0, y_0)$  be the intersection point between the graphs of  $f(x)$  and  $g(x)$ .



- a. [6 points] Compute  $P(x)$ , the function containing the first three nonzero terms of the Taylor series about  $x = 0$  of  $f(x) = 6 \cos(\sqrt{2x})$ .
- b. [3 points] Use  $P(x)$  to approximate the value of  $x_0$ .
- c. [3 points] Use  $P(x)$  and the value of  $x_0$  you computed in the previous question to write an integral that approximates the value of the shaded area. Find the value of this integral.
- d. [1 point] Graph  $f(x)$  and  $g(x)$  in your calculator. Use the graphs to find an approximate value for  $x_0$ .
- e. [2 points] Write a definite integral in terms of  $f(x)$  and  $g(x)$  that represents the value of the shaded area. Find its value using your calculator.