Math 116 — First Midterm

October 13, 2010

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	12	
2	11	
3	12	
4	13	
5	13	
6	12	
7	12	
8	15	
Total	100	

1. [12 points] Indicate if each of the following is true or false by circling the correct answer (Justify your answer):

a. [3 points] If $F(t)$ and $G(t)$ are antiderivatives of the function $f(t)$ with $F(0) = 1$ and $G(0) = 3$ then $F(2) - G(2) = 1$.

True **[False]**

Solution: $F(2) - G(2) = F(0) - G(0) = 1 - 3 = -2$

b. [3 points] If $h(t) > 0$ for $0 \leq t \leq 1$, then the function $H(x) = \int_0^x h(t)dt$ is concave up for $0 \leq x \leq 1$.

True **[False]**

Solution: $H'(x) = h(x) > 0$, hence $H(x)$ is increasing in $0 \leq x \leq 1$, but not necessarily concave up. You need $H''(x) = h'(x) > 0$ for $H(x)$ to be concave up.

c. [3 points] If $\int_0^2 g(t)dt = 6$ then $\int_2^3 3g(2t-4)dt = 9$.

[True] False

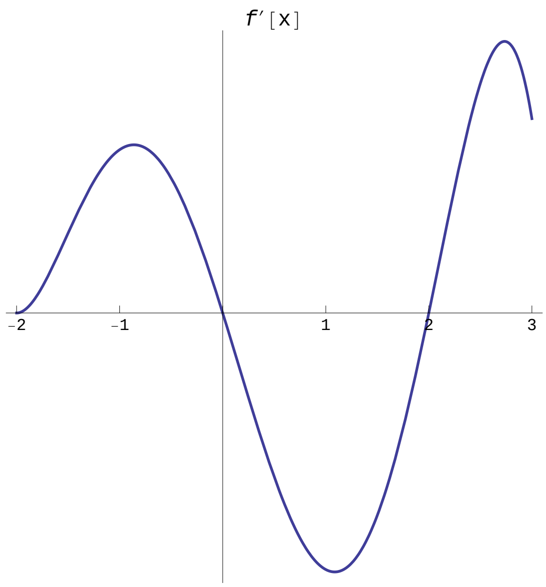
Solution: Using $u = 2t - 4$ then $\int_2^3 3g(2t-4)dt = \frac{3}{2}\int_0^2 g(u)du = 9$

d. [3 points] $\frac{d}{dx}\left(\int_{-x^2}^{\sin x} e^{t^3}dt\right) = \cos x\ e^{x^3} + 2xe^{x^3}$.

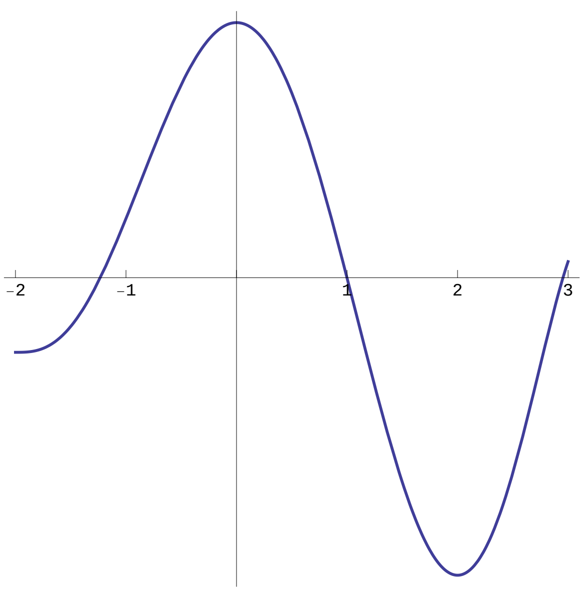
True **[False]**

Solution: $\frac{d}{dx}\left(\int_{-x^2}^{\sin x} e^{t^3}dt\right) = \cos x\ e^{\sin^3 x} + 2xe^{-x^6}$.

2. [11 points] Given the graph of $f'(x)$. Sketch a graph of $f(x)$ on the provided axes given that $f(1) = 0$. On your graph, label any local maxima, minima, and points of inflection. Make sure that the concavity of the graph of $f(x)$ is visible in your graph.



Solution:
$x = 1,$ x-intecept
$x = 0$ local max
$x = 2$ local min
$x = -1, 1, 2.8$ points of inflections
$(-2, -1)$, $(1, 2.8)$ concave up
$(-1, 1)$,$(2.8, 3)$ concave down



3. [12 points]

a. [8 points] Estimate the value of I, where

$$I = \int_0^1 e^{-\frac{t^2}{2}}\,dt$$

using $LEFT(3)$, $RIGHT(3)$, $MID(3)$ and $TRAP(3)$. Write each sum.

Solution:

t	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$e^{-\frac{t^2}{2}}$	1	.9862	.946	.8825	.8007	.7006	.6065

$LEFT(3) = \frac{1}{3}(1 + .946 + .8007) = .9156$

$RIGHT(3) = \frac{1}{3}(.946 + .8007 + .6065) = .7844$

$TRAP(3) = \frac{.9156+.7844}{2} = .85$

$MID(3) = \frac{1}{3}(.9862 + .8825 + .7006) = .8564.$

b. [4 points] Which among the four Riemann sums ($LEFT(n)$, $RIGHT(n)$, $MID(n)$ and $TRAP(n)$) yields the closest underestimate to I, for any number n of subdivisions of the interval $[0, 1]$? Justify your answer.

Solution: $f(t) = e^{-\frac{t^2}{2}}$

$f'(t) = -te^{-\frac{t^2}{2}} < 0$ on $t \in (0, 1]$

$f''(t) = e^{-\frac{t^2}{2}}(t^2 - 1) \leq 0$ on $t \in [0, 1]$.

Since $f(t)$ is concave down and decreasing in $[0, 1)$, then $TRAP(n)$ yields the closest underestimate for I for all n.

4. [13 points]

a. [6 points] Compute

$$\int_0^1 f'(x)\sin(2\pi x)dx$$

where $f(x)$ is given by the following table.

x	0	.25	.5	.75	1
$f(x)$	.8	.5	1.2	1	.4

Solution:

$$\int_0^1 f'(x)\sin(2\pi x)dx = f(x)\sin(2\pi x)\,\Big|_0^1 - 2\pi\int_0^1 f(x)\cos(2\pi x)dx$$
$$= -2\pi\int_0^1 f(x)\cos(2\pi x)dx$$

x	0	.25	.5	.75	1
$f(x)\cos(2\pi x)$	.8	0	-1.2	0	.4

$$\begin{aligned}\text{left sum} &= .6283\\ \text{right sum} &= 1.2566\\ \text{trapezoid sum} &= .9424\end{aligned}$$

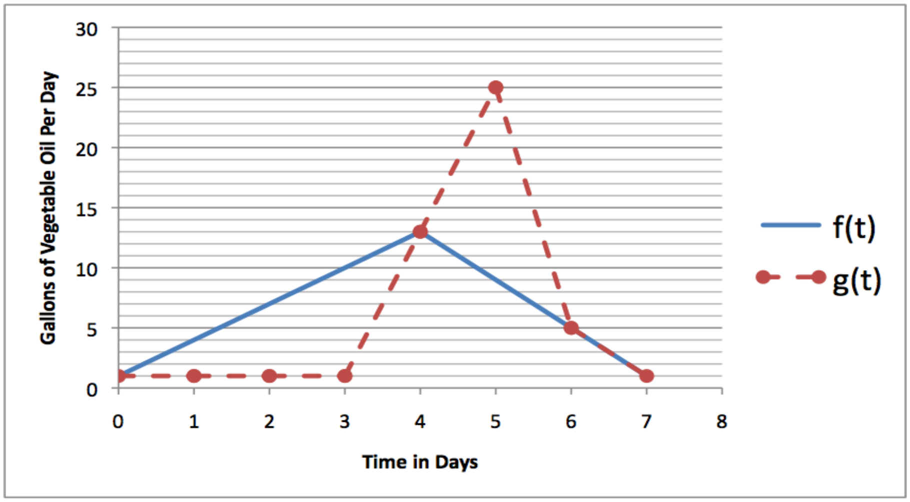
b. [7 points] Find

$$\int x^3\cos x^2 dx.$$

Solution:

$$\begin{aligned}
&\int x^3\cos x^2 dx = \\
u = x^2 \qquad &= \frac{1}{2}\int u\cos u\,du\\
\text{by parts} \qquad &= \frac{1}{2}\left(u\sin u - \int \sin u\,du\right)\\
&= \frac{1}{2}\left(u\sin u + \cos u\right) + C\\
&= \frac{1}{2}\left(x^2\sin x^2 + \cos x^2\right) + C
\end{aligned}$$

5. [13 points] In 2008, the burrito chain BTB began to operate a "Party Bus" powered by waste vegetable oil. If t is the number of days since 12:01 a.m. on October 11, 2010, then $f(t)$ is the amount in gallons per day of waste vegetable oil produced by BTB restaurant chain at time t and $g(t)$ is the amount consumed by the party bus in gallons per day at time t. Let $R(t)$ be the size of BTB's vegetable oil reserves in gallons at time t. If BTB has 20 gallons held in reserve at time $t = 0$, use the graphs below to answer the following questions. All the questions below consider only $0 \le t \le 7$.



a. [1 point] Estimate $R(3)$

Solution: $R(3) = 20 + \int_0^3 f(t) - g(t)dt = 33.5$ gallons

b. [2 points] When does BTB have a maximum volume of vegetable oil in reserve?

Solution: After 4 days (Oct 15).

c. [3 points] Suppose you need a ride to the airport on October 16. Will BTB have any vegetable oil in reserve to power their bus and drive you to the airport that day?

Solution: $R(5) = 20 + \int_0^5 R(t)dt = 29$ gallons

d. [3 points] Find all critical points of $R(t)$.

Solution: Critical points: $t = 4$ and all $6 \le t < 7$.

e. [4 points] On what intervals is $R(t)$ concave up? On what intervals is $R(t)$ concave down?

Solution:
$R''(t) > 0$ on $(0,3)$, $(5,6)$ then concave up
$R''(t) < 0$ on $(3,4)$,$(4,5)$ then concave down.

6. [12 points]

a. [8 points] Robotic submersibles are often used to maintain deep sea oil wells, and the recent BP spill inspired Trevion to design his own. His submersible will have the shape of a cube 1.5 meter in length. If the bottom of the submersible is 800 meters below the surface of the ocean, what is the force exerted by the water on each of the submersible's six outer surfaces? (Water density is $1000\frac{kg}{m^3}$).

Solution:
Top surface: $F = (1000\frac{kg}{m^3})(9.8\frac{m}{s^2})(798.5m)(1.5m)^2 = 17,606,925$ Newtons.

Bottom surface: $F = (1000\frac{kg}{m^3})(9.8\frac{m}{s^2})(800m)(1.5m)^2 = 17,640,000$ Newtons.

Side surfaces: $F = \int_{798.5}^{800} 1000(9.8)(1.5)h dh = 7350h^2 \big|_{798.5}^{800} = 17,623,462.5$ Newtons

h =distance from the level of the sea.

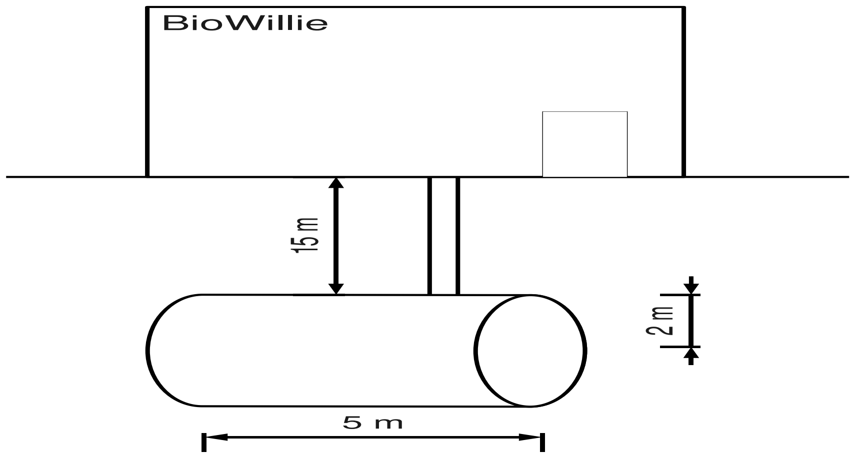
b. [4 points] Let S be the solid generated by rotating the area enclosed by the curves $y = x^2$ and $y = \sqrt{x}$ around the line $x = 1$. Set up, but do not evaluate a definite integral that can be used to compute the volume of S.

Solution:

$$\begin{aligned} V &= \int_0^1 \pi(r_1^2 - r_2^2)dy \\ &= \int_0^1 \pi((1-y^2)^2 - (1-\sqrt{y})^2)dy \end{aligned}$$

where
r_1 = horizontal distance between the line $x = 1$ and the graph $y = \sqrt{x}, (x = y^2)$
r_2 = horizontal distance between the line $x = 1$ and the graph $y = x^2, (x = \sqrt{y})$

7. [12 points] Country music legend Willie Nelson is concerned about our dependence of fossil fuels. In 2005, he started a company which sells a bio-diesel fuel called BioWillie. He recently added a new cylindrical underground storage tank at his factory, and he needs to know how much work is required to pump all the fuel in a full tank to the surface. The tank is pictured below. It is 5 meters long and has a radius of 2 meters. Its center line is 17 meters underground. BioWillie fuel has a density of 900 kg per cubic meter. Make sure to include appropriate units and justification to support your answers.



a. [7 points] Write an expression that approximates the work done in lifting a horizontal slice of fuel that is h_i meters below the ground's surface, given that the thickness of the slice is Δh meters.

Solution: Using the ground's surface as our horizontal axis and h be the variable on the vertical axis, then the cross section of the tank can be described by $x^2+(h+17)^2=4$.

Volume$_{slice} = 2\sqrt{4-(h+17)^2}(5)\Delta h\ m^3$.

Force$_{slice}$ = (density)(Volume$_{slice}$)g Newtons

Distance= $h_i = -h$ m.

Work$_{slice}$ =(Force)(Distance)= $(900)(2\sqrt{4-(h+17)^2}(5)\Delta h)g(-h)$ Joules.

b. [5 points] Help Willie Nelson by computing the work required to pump all the fuel in a full tank to the ground's surface. You can use your calculator to compute your final answer.

Solution: $Work = \int_{-19}^{-15} -9000\sqrt{4-(h+17)^2}ghdh = 9,421,008.05$ Joules.

8. [15 points] A metal plate with constant density $5\frac{kg}{m^2}$ has a shape bounded by the curve $y = 1 - x^2$ and the x-axis. You should evaluate all definite integrals in this problem by hand.

a. [3 points] Write an expression that approximates the amount of mass contained in a horizontal slice of the metal plate of width Δy meters, located at y_i meters from the x axis. Your answer should be be in terms of y_i and Δy.

Solution:

m_i =(density) (area of slice)= $(5)(2\sqrt{1-y_i}\Delta y)$.

b. [4 points] Find the exact total mass of the metal plate.

Solution:

$$m = \int_0^1 10\sqrt{1-y}dy = -\frac{20}{3}(1-y)^{\frac{3}{2}} \,|_0^1 = \frac{20}{3} = 6.66 \text{ kg}.$$

c. [8 points] Find the center of mass of the metal plate.

Solution:

$\bar{x}$=**0** by symmetry.

$$\bar{y} = \frac{\int_0^1 10y\sqrt{1-y}dy}{\frac{20}{3}} = \frac{3}{2}\int_0^1 y\sqrt{1-y}dy = \frac{3}{2}\left(\frac{4}{15}\right) = \frac{2}{5} = .4$$

With u substitution:

$$\int_0^1 y\sqrt{1-y}dy =$$

$$u = 1-y \qquad = -\int_1^0 (1-u)u^{\frac{1}{2}}du$$

$$= \frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}} \,|_0^1 = \frac{4}{15}$$

With integration by parts:

$$\int_0^1 y\sqrt{1-y}dy = -\frac{2}{3}y\sqrt{1-y}|_0^1 + \frac{2}{3}\int_0^1 (1-y)^{\frac{3}{2}}dy$$

$$= -\frac{2}{3}\left(\frac{2}{5}(1-y)^{\frac{5}{2}}\right) |_0^1 = \frac{4}{15}$$