Math 116 — Second Midterm

November 16, 2011

Name: ____

Instructor: ______________________________ Section: ______________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	12	
2	12	
3	11	
4	11	
5	11	
6	8	
7	13	
8	10	
9	12	
Total	100	

1. [12 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

a. [2 points] The function $y(t) = \cos 3t + B \sin 3t + \frac{1}{9}t$ is a solution of $y'' + 9y = 0$ with $y(0) = 1$.

True False

b. [2 points] The value of the integral used to compute the area enclosed by a curve $r = f(\theta)$ given in polar coordinates can be negative if $f(\theta) \leq 0$.

True False

c. [2 points] If $f(x)$ is a continuous function such that $\int_1^\infty f(x)dx$ diverges, then $\int_1^\infty f(x)^2 dx$ must diverge.

True False

d. [2 points] If $P(x)$ is a cumulative distribution function for the probability density function $p(x)$, then $1 + P(x)$ is also a cumulative distribution function for $p(x)$.

True False

e. [2 points] All solutions to the differential equation $y' = 1 + y^4$ are increasing functions.

True False

f. [2 points] Let $P(t)$ be the cumulative distribution function of a probability density function $p(t)$. If $P(0) = \frac{2}{3}$ then the median of $p(t)$ is negative.

True False

2. [12 points] Consider a particle whose trajectory in the xy-plane is given by the parametric curve defined by the equations

$$x(t) = t^4 - 4t^2, \qquad y(t) = t^2 - 2t,$$

for $-3 \leq t \leq 3$. Show all your work to receive full credit.

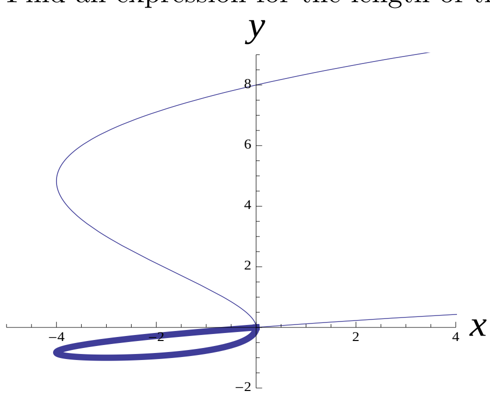
a. [3 points] Is there any value of t at which the particle ever comes to a stop? Justify.

b. [2 points] For what values of t does the path of the particle have a vertical tangent line?

c. [3 points] What is the lowest point (x, y) on the curve?

d. [2 points] At what values of t does the particle pass through the origin?

e. [2 points] The graph of the curve traced by these parametric equations is shown below. Find an expression for the length of the closed loop marked in the graph.



3. [11 points] In the late 30th century, Mom's Friendly Robot Company is the main global robot manufacturing company. The Bending Unit 22 model is designed to contain a backup unit, effectively rendering it immortal. However, a small percentage of the robots suffer a manufacturing defect, in which the backup unit is malfunctional or not present. The function

$$p(t) = \begin{cases} 0, & \text{if } t < 0 \\ 0.004e^{-t/c}, & \text{if } t \geq 0 \end{cases}$$

gives the probability density for the lifetime of these defective Bending Units 22, where c is a positive constant and t is measured in years since the robots are activated. Show all your work to receive full credit.

a. [2 points] Interpret the quantity $\displaystyle\int_{100}^{140} p(t)\, dt$.

b. [4 points] Find the value of c.

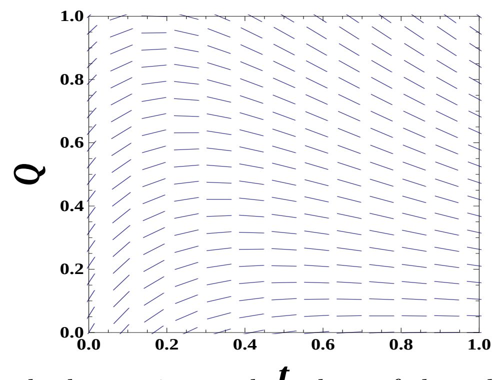
c. [5 points] What is the mean (average) lifespan of a defective Bending Unit 22?

4. [11 points] A restaurant installs a kitchen ventilation system to control the amount of grease in the air due to cooking. The ventilation system reduces the amount of grease in the air by 90 percent every hour. Let $Q(t)$ be the amount in grams of grease in the air t hours after the ventilation is activated. Then Q satisfies the differential equation

$$\frac{dQ}{dt} = 2e^{-5t} - \frac{9}{10}Q,$$

where $2e^{-5t}$ is the rate at which the kitchen produces grease in grams per hour at time t.

a. [2 points] The slope field of the differential equation is shown below. Suppose that the air in the kitchen initially has 0.4 grams of grease. Sketch the solution curve in the slope field.



b. [7 points] Use Euler's method to estimate the values of the solution curve $Q(t)$ through $(0, 0.4)$ for all values of t given in the table below. Show all your work.

t	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$Q(t)$				

c. [2 points] Does your approximation for $Q(1)$ using Euler's method give an overestimate or an underestimate? Justify.

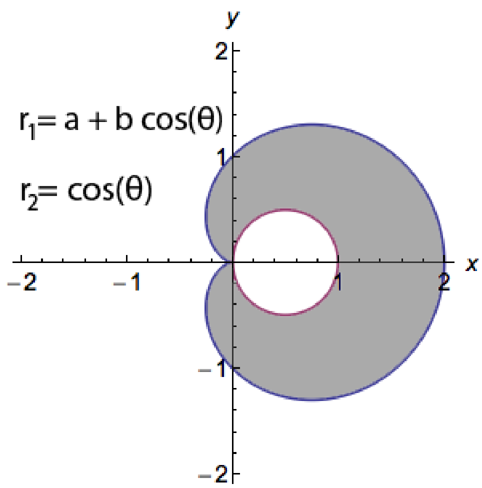
5. [11 points] Determine if the following integrals converge or diverge. If an integral diverges, explain why. If it converges, find the value to which it converges. Mathematical precision is important.

a. [4 points] $\displaystyle\int_{-1}^{2} \frac{1}{\sqrt{2-x}}\, dx$

b. [4 points] $\displaystyle\int_{10}^{\infty} \frac{5+2\sin(4\theta)}{\theta}\, d\theta$

c. [3 points] $\displaystyle\int_{1}^{\infty} \frac{x}{1+x}\, dx$

6. [8 points] Members of the recruitment committee for the Mars University (MU) chapter of the fraternity Epsilon Rho Rho are designing a pledge pin to distribute during Rush Week. The pin takes the shape of a cardioid with a circular hole in it. The cardioid is given by a polar equation of the form $r_1 = a + b\cos(\theta)$, while the circular hole has the polar equation $r_2 = \cos(\theta)$. The pin is pictured below, where the x- and y-axes are measured in inches.



a. [5 points] The committee plans on coating one side of the pin in gold plating, which costs 3 dollars per square inch. Give an expression representing the cost to plate one face of the pin in gold. Your answer may involve integrals and the constants a and b.

b. [3 points] Find a and b. $a =$ ________ $b =$ ________

7. [13 points] A company designs an air filter for a ship's engine room that reduces the amount of fumes in the air by k percent every hour. The machinery in the engine room produces fumes at a rate of 0.02 kilograms per hour. Let $Q(t)$ be the amount in kilograms of fumes in the room t hours after the engines are activated. Hence Q satisfies

$$\frac{dQ}{dt} = 0.02 - \frac{k}{100}Q.$$

a. [9 points] Find a formula for $Q(t)$. Suppose there are no fumes in the air when the engines are activated.

b. [2 points] What is the value of $Q(t)$ in the long run?

c. [2 points] Air safety regulations require that the *concentration* of fumes in the air not exceed 10^{-4} kilograms per liter at any time. If the volume of air in the engine room is 10^3 liters, for what values of k are the safety regulations met at all times?

8. [10 points] For $\alpha > 0$, consider the family of spirals given by $r = \frac{1}{\theta^\alpha}$ in polar coordinates.

a. [2 points] Write down an integral that gives the length L of a spiral in this family for $1 \le \theta \le b$. No credit will be given if you just write down the formula given in part (b).

b. [8 points] It can be shown that the length L of the spiral in part a) may also be written as

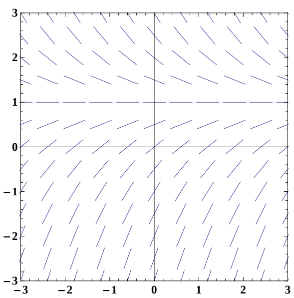
$$L = \int_1^b \frac{1}{\theta^\alpha}\sqrt{1 + \frac{\alpha^2}{\theta^2}}\, d\theta.$$

Use this formula for L to find all values of $\alpha > 0$ for which the length of the spiral is infinite for $1 \le \theta$. For which values of α is the length finite? Justify all your answers using the comparison test.

9. [12 points] Match each of the following differential equations with its possible slope field. Circle your answers. No justification is required.

(a) $y' = ay$ with $a > 0$. I II III IV

(b) $y' = k(y - a)$ with $a > 0$ and $k < 0$. I II III IV



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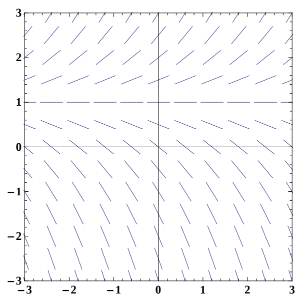

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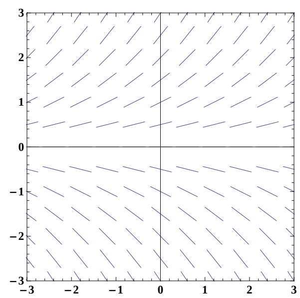

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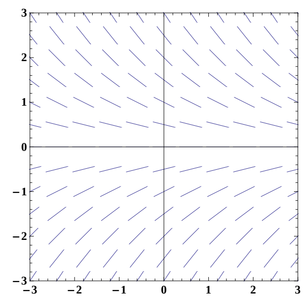
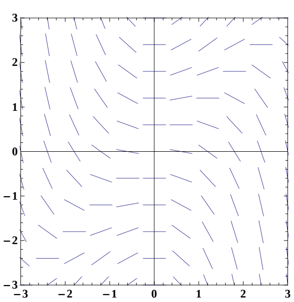


IV

(a) $y' = x(y - ax)$ with $a > 0$. V VI VII VIII

(b) $y' = ay^2 - y$ with $a > 0$. V VI VII VIII



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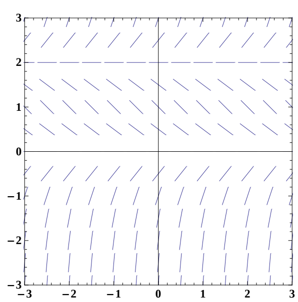

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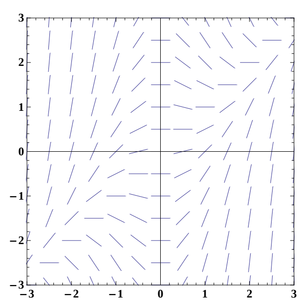
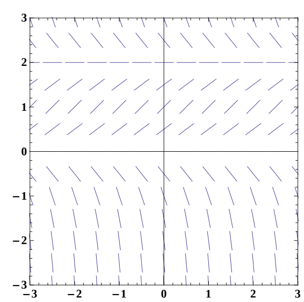


VII



VIII