# Math 116 — First Midterm

October 12, 2011

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

| Problem | Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 12 | |
| 3 | 9 | |
| 4 | 12 | |
| 5 | 11 | |
| 6 | 14 | |
| 7 | 13 | |
| 8 | 8 | |
| 9 | 9 | |
| Total | 100 | |

**1**. [12 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

**a**. [2 points] If $f$ and $g$ are continuous functions over the interval $[a, b]$, then the average value of $f(x)g(x)$ over that interval is the average value of $f$ times the average value of $g$ over that interval.

True $\boxed{\text{False}}$

*Solution:*

**b**. [2 points] The units of $\int f(x)dx$ are the same as the units of $f(x)$.

True $\boxed{\text{False}}$

*Solution:*

**c**. [2 points] If $f(x)$ is even and $\int_0^2 f(x)dx = 3$, then $\int_{-2}^2 (f(x) - 4)dx = -10$.

$\boxed{\text{True}}$ False

**d**. [2 points] The center of mass of an object can be outside of the object.

$\boxed{\text{True}}$ False

*Solution:*

**e**. [2 points] Over the interval $[0, 1]$, if LEFT(2) = RIGHT(2) for a continuous function $f(x)$, then we know

$$\text{LEFT}(2) = \int_0^1 f(x)dx = \text{RIGHT}(2).$$

True $\boxed{\text{False}}$

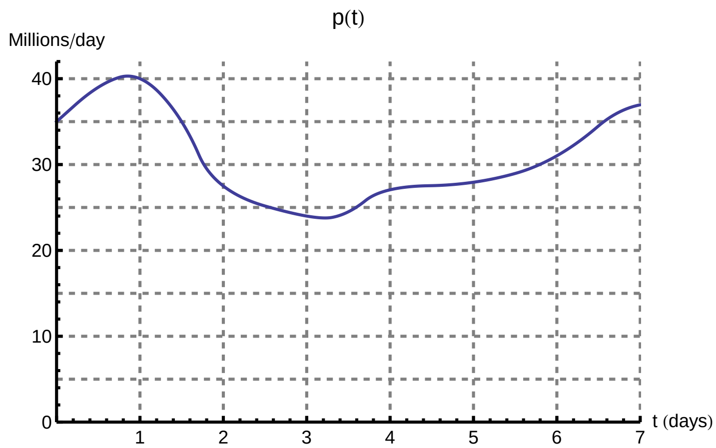
*Solution:*

**f**. [2 points] Let $f(x) > 0$ be a continuous function. Then $F(x) = \int_0^x f(t)dt \geq 0$ for all values of $x$.

True $\boxed{\text{False}}$

*Solution:*

**2**. [12 points] Photo sharing through social networking sites has become increasingly popular over the years. Suppose $p(t)$ gives the rate at which photos are uploaded to Facebook's servers, over a certain one-week period, in millions of photos per day. ($t = 0$ corresponds to the beginning of Sunday.) A graph of $p(t)$ is given below.




**a**. [2 points] Write a definite integral that gives the total number of photos uploaded to Facebook from the beginning of Sunday through the end of Monday. Include units in your answer.

*Solution:* $\int_0^2 p(t)dt$ millions of photos *or* $10^6 \int_0^2 p(t)dt$ photos

**b**. [8 points] Estimate the value of the definite integral in part (a) using LEFT(2), RIGHT(2), MID(2) and TRAP(2). Write each sum in terms of $p$.

*Solution:* $\Delta t = \frac{2-0}{2} = 1$, so the width of each rectangle in the Riemann sums is 1. Then

$$\text{LEFT}(2) = p(0)\Delta t + p(1)\Delta t \approx 35 \cdot 1 + 40 \cdot 1 = 75 \text{ million photos}$$

$$\text{RIGHT}(2) = p(1)\Delta t + p(2)\Delta t = 40 \cdot 1 + 27.5 \cdot 1 \approx 67.5 \text{ million photos}$$

$$\text{MID}(2) = p(0.5)\Delta t + p(1.5)\Delta t = 38 \cdot 1 + 35 \cdot 1 \approx 73 \text{ million photos}$$

$$\text{TRAP}(2) = \frac{\text{LEFT}(2) + \text{RIGHT}(2)}{2} \approx \frac{75 + 67.5}{2} = 71.25 \text{ million photos}$$

In terms of the function $p$, one can also calculate

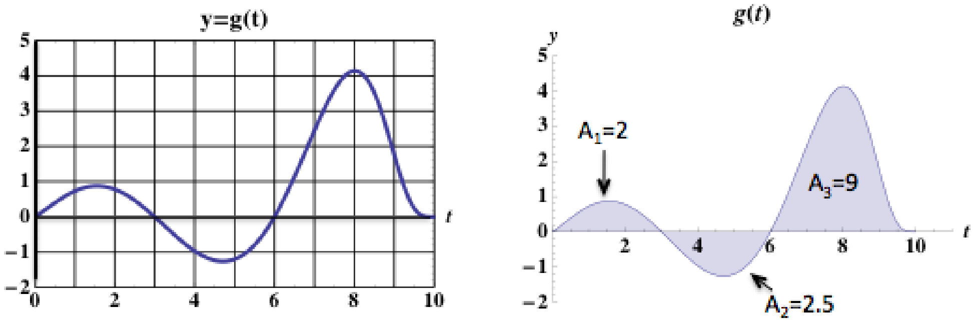
$$\text{TRAP}(2) = \frac{1}{2}p(0) + p(1) + \frac{1}{2}p(2) \approx \frac{1}{2} \cdot 35 + 40 + \frac{1}{2} \cdot 27.5 = 71.25 \text{ million photos.}$$

Note that answers may vary slightly because exact values of the $p(t)$ were not given in the statement of the problem.

**c**. [2 points] Give a real world interpretation of the quantity $\frac{1}{5}\int_1^6 p(t)dt$. Include units.

*Solution:* Since $\frac{1}{5}\int_1^6 p(t)dt = \frac{1}{6-1}\int_1^6 p(t)$, the given integral represents the average rate at which photos are uploaded to Facebook's servers during the work week (Monday through Friday).

**3**. [9 points] The graph of $g(t)$ and the areas $A_1$, $A_2$ and $A_3$ between its graph and the $t$ axis are shown below.




Let

$$H(x) = \int_3^{3x} g(t)dt \qquad \text{and} \qquad F(x) = \int_0^x g(t)dt.$$

**a**. [5 points] Find $H(1)$, $H(2)$ and $H'(3)$.

*Solution:*
$H(1) = \int_3^3 g(t)dt = 0$ $\qquad H(2) = \int_3^6 g(t)dt = -2.5$ $\qquad H'(3) = 3g(9) = 6.$

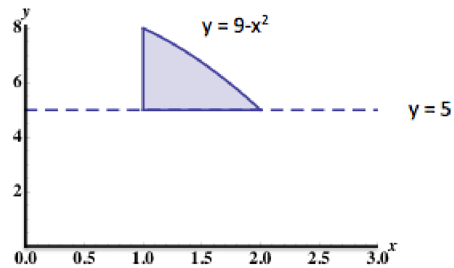
**b**. [2 points] For what values of $5 \le x \le 10$ is $F(x)$ increasing?

*Solution:* $6 \le x \le 10.$

**c**. [2 points] For what values of $5 \le x \le 10$ is $F(x)$ concave up?

*Solution:* $5 \le x \le 8.$

**4.** [12 points] Consider the region in the $xy$-plane bounded by the curves $y = 9 - x^2, x = 1$, and $y = 5$. This region is pictured below.




Give a definite integral that computes the quantities below. You do not need to evaluate these integrals.

**a.** [3 points] The area of the region shown.

*Solution:*
A$= \int_1^2 (9-x^2) - 5dx = \int_1^2 4 - x^2 dx.$

A$= \int_5^8 \sqrt{9-y} - 1dy.$

**b.** [3 points] The volume of the solid obtained by rotating the region about the $y$-axis.

*Solution:*

$$V = \int_5^8 \pi \left( \left(\sqrt{9-y}\right)^2 - 1 \right) dy \quad \text{or} \quad V = \int_1^2 2\pi x \left(9 - x^2 - 5\right) dx = \int_1^2 2\pi x \left(4 - x^2\right) dx$$

**c.** [3 points] The volume of the solid obtained by rotating the region about the $x$-axis.

*Solution:*

$$V = \int_1^2 \pi \left( \left(9 - x^2\right)^2 - 25 \right) dx \quad \text{or} \quad V = \int_5^8 2\pi y \left( \sqrt{9-y} - 1 \right) dy$$

**d.** [3 points] The volume of the solid obtained by rotating the region about the line $y = 5$.

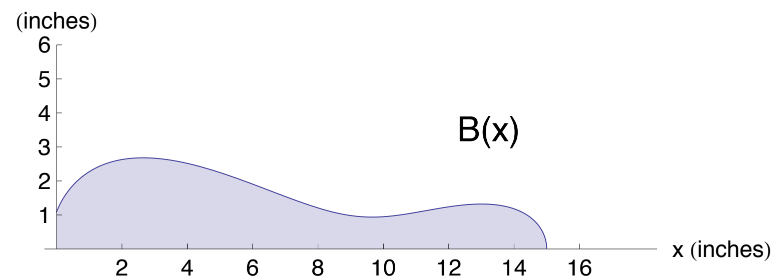
*Solution:*

$$V = \int_1^2 \pi \left(4 - x^2\right)^2 dx \quad \text{or} \quad V = \int_5^8 2\pi(y-5) \left( \sqrt{9-y} - 1 \right) dy$$

**5**. [11 points] During a friendly game of ten-pin bowling, your friends Walter and Smokey begin to quarrel over whether Smokey's toe slipped over the foul line. Meanwhile, you decide to pass the time by finding a mathematical model for the shape of a bowling pin. After some careful thought, you find that a fallen pin is a solid of revolution given by rotating the region under the curve

$$B(x) = \sqrt{1.2 + 5.32x - 1.485x^2 + .135x^3 - .004x^4}$$

over the interval $[0, 15]$ about the $x$-axis. The region is pictured below. All measurements are in inches. A helpful stranger in the bowling alley informs you that the wood used to make the pin has density $\delta = 17$ grams per cubic inch.




**a**. [3 points] Write a definite integral that gives the mass of the bowling pin. You do not need to evaluate this integral.

*Solution:* Since the bowling pin is a solid of revolution, the volume of a cylindrical slice located $x$ inches from the base of the pin can be approximated by $\pi B(x)^2 \Delta x$. Thus, the mass of the slice is approximately

$$\delta\pi B(x)^2\Delta x = 17\pi\left(\sqrt{1.2 + 5.32x - 1.485x^2 + .135x^3 - .004x^4}\right)^2 \Delta x,$$

so that the mass of the entire pin is

$$\int_0^{15} 17\pi B(x)^2 dx = \int_0^{15} 17\pi\left(1.2 + 5.32x - 1.485x^2 + .135x^3 - .004x^4\right) dx \text{ grams.}$$

**b**. [6 points] What are the coordinates $(\bar{x}, \bar{y})$ of the bowling pin's center of mass? You may use your calculator to answer this question.

*Solution:* Since the bowling pin has uniform density, we know immediately from rotational symmetry that $\bar{y} = 0$. Using the formula for the $x$-coordinate of the center of mass, we have

$$\begin{aligned}\bar{x} &= \frac{\int_0^{15} \delta x\pi B(x)^2 dx}{\int_0^{15} \delta\pi B(x)^2 dx} \\ &= \frac{\int_0^{15} 17x\pi\left(1.2 + 5.32x - 1.485x^2 + .135x^3 - .004x^4\right) dx}{\int_0^{15} 17\pi\left(1.2 + 5.32x - 1.485x^2 + .135x^3 - .004x^4\right) dx} = 5.\end{aligned}$$

Thus, the coordinates of the center of mass of the bowling pin are $(5, 0)$, where each coordinate is measured in inches.

**c.** [2 points] Suppose the wood used to make the pin had density $\delta = 16$ grams per cubic inch. How does this affect the position $(\bar{x}, \bar{y})$ of the center of mass?

*Solution:* The center of mass is not affected since the integral

$$\bar{x} = \frac{\int_0^{15} \delta x \pi B(x)^2 dx}{\int_0^{15} \delta \pi B(x)^2 dx} = \frac{\int_0^{15} x \pi B(x)^2 dx}{\int_0^{15} \pi B(x)^2 dx}$$

is independent of $\delta$. The same is true for $\bar{y}$.

**6.** [14 points] A botanical garden has the shape of the region in the $xy$-plane bounded by the curve $y = x^2$ and the $x$-axis, with $0 \le x \le 8$. One of the responsibilities of the gardener, is to keep the garden free of a poisonous weed. The density $\delta$ of the weed at any point in the garden depends on the distance $x$ from the $y$-axis. Values of $\delta$ are given in kg of plants per meter square in the table below.

| $x$ (meters) | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| $\delta(x)$ | 10 | 12 | 15 | 17 | 18 |

**a.** [3 points] Write an integral that computes the total amount of weed in the garden. Include units.

*Solution:* $\int_0^8 x^2\delta(x)dx$ kg .

**b.** [3 points] Compute RIGHT(4) for the integral in (a). Write out all the terms in the sum. Does this sum give an overestimate or an underestimate for the total amount of weed in the garden? Justify.

*Solution:* $\int_0^8 x^2\delta(x)dx \approx 2(2^2 \cdot 12 + 4^2 \cdot 15 + 6^2 \cdot 17 + 8^2 \cdot 18) = 4104$ .

RIGHT(4) is an overestimate because $\delta(x)$ is increasing and $x^2$ is increasing then $x^2\delta(x)$ is increasing.

**c.** [2 points] Which of the following approximations to (a) are computable with the given data? Circle all that apply.

MID(1) MID(2) MID(3) MID(4)

*Solution:*

<u>MID(1)</u> <u>MID(2)</u> MID(3) MID(4)

**d.** [1 point] Which Riemann sum gives the best estimate for the integral in (a)? Circle one.

RIGHT(4) LEFT(4) TRAP(4)

*Solution:*

RIGHT(4) LEFT(4) <u>TRAP(4)</u>

**e.** [5 points] The gardener built a fence around the garden. How long is the fence? Include units. You may use your calculator.

Length of the fence $= 8+64+\int_0^8 \sqrt{1+\left(\frac{d}{dx}(x^2)\right)^2}dx = 72+\int_0^8 \sqrt{1+4x^2}dx = 136.99$ meters.

**7**. [13 points] Household electricity in the United States is supplied in the form of an alternating current that varies sinusoidally with a frequency of 60 cycles per second (Hz). The voltage is given by the equation

$$E(t) = 170\sin(120\pi t),$$

where $t$ is given in seconds and $E$ is in volts.

**a**. [7 points] Using integration by parts, find $\int \sin^2\theta d\theta$. Show all work to receive full credit. (Hint: $\sin^2\theta + \cos^2\theta = 1$.)

*Solution:* We first note that $\int \sin^2\theta d\theta = \int \sin\theta(\sin\theta)d\theta$, so that we may take $u = \sin\theta, dv = \sin\theta d\theta$ (and $du = \cos\theta d\theta, v = -\cos\theta$). Then integration by parts gives

$$\begin{aligned}\int \sin^2\theta d\theta &= \sin\theta(-\cos\theta) - \int -\cos\theta(\cos\theta)d\theta \\ &= -\sin\theta\cos\theta + \int \cos^2\theta d\theta.\end{aligned}$$

Using the trig. identity given in the hint, we obtain

$$\int \sin^2\theta d\theta = -\sin\theta\cos\theta + \int \left(1 - \sin^2\theta\right) d\theta = -\sin\theta\cos\theta + \int d\theta - \int \sin^2\theta.$$

The integral on the far right also appears on the left, so combining like terms, we get

$$\begin{aligned}2\int \sin^2\theta d\theta &= -\sin\theta\cos\theta + \int d\theta \\ \int \sin^2\theta d\theta &= \frac{1}{2}\left(-\sin\theta\cos\theta + \int d\theta\right) \\ &= -\frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + C.\end{aligned}$$

**b**. [6 points] Voltmeters read the root-mean-square (RMS) voltage, which is defined to be the square root of the average value of $[E(t)]^2$ over one cycle. Find the exact RMS voltage of household current.

*Solution:* Since the frequency of the current is 60 cycles per second, one cycle is completed every $\frac{1}{60}$ seconds. Thus

$$\begin{aligned}\text{RMS voltage} &= \sqrt{\frac{1}{\frac{1}{60}-0}\int_0^{\frac{1}{60}} E(t)^2 dt}\\ &= \sqrt{60\int_0^{\frac{1}{60}} 170^2 \sin^2(120\pi t)\,dt}.\end{aligned}$$

Substituting $w = 120\pi t, dw = 120\pi dt$, we get

$$\begin{aligned}\text{RMS voltage} &= \sqrt{60\int_{w(0)}^{w(\frac{1}{60})} 170^2 \sin^2(w)\cdot\frac{1}{120\pi}dw}\\ &= \sqrt{\frac{170^2}{2\pi}\int_0^{2\pi}\sin^2(w)\,dw}.\end{aligned}$$

Using the antiderivative we found in part (a) with $C = 0$, the Fundamental Theorem of Calculus gives

$$\begin{aligned}\text{RMS voltage} &= \sqrt{\frac{170^2}{2\pi}\left(-\frac{1}{2}\sin\theta\cos\theta+\frac{\theta}{2}\right)\Big|_0^{2\pi}}\\ &= \sqrt{\frac{170^2}{2\pi}\left((-\frac{1}{2}\sin(2\pi)\cos(2\pi)+\frac{2\pi}{2})-(-\frac{1}{2}\sin(0)\cos(0)+\frac{0}{2})\right)} = \frac{170}{\sqrt{2}}\text{ Volts.}\end{aligned}$$

Note that due to the periodicity of the sine function, the average value over one cycle could also have been computed over $0 \le t \le 1$ (or any other number of periods).

**8**. [8 points] Let $f$ be a differentiable function with derivative $f'$. A table of values for $f$ and $f'$ is given below.

| $t$ | 0 | 3 | 6 | 9 |
|---|---|---|---|---|
| $f(t)$ | 1 | 2 | 7 | 5 |
| $f'(t)$ | 1 | 4 | -1 | -2 |

Find the exact value of the following integrals.

**a**. [3 points] $\int_0^1 f'(3t)dt$.

*Solution:* Substituting $w = 3t, dw = 3dt$, we obtain

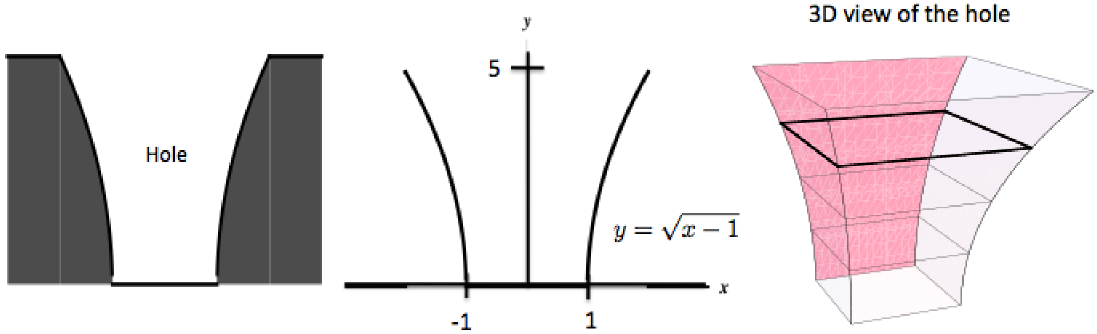
$$\int_{w(0)}^{w(1)} f'(w) \cdot \frac{1}{3} dw = \frac{1}{3}\int_0^3 f'(w)dw = \frac{1}{3}\left(f(3) - f(0)\right) = \frac{1}{3}(2-1) = \frac{1}{3}.$$

**b**. [5 points] $\int_3^9 tf''(t)dt$.

*Solution:* Using integration by parts with $u = t, dv = f''(t)dt$ (so that $du = dt, v = f'(t)$), we obtain

$$\begin{aligned}\int_3^9 tf''(t)dt = t\cdot f'(t)\Big|_3^9 - \int_3^9 f'(t)dt &= \big(9f'(9) - 3f'(3)\big) - (f(9) - f(3))\\ &= (9\cdot(-2) - 3\cdot 4) - (5-2) = -33.\end{aligned}$$

**9**. [9 points] As part of an exploration assignment, a team of mining engineers dug a hole in the ground. The hole takes the shape of a solid region of known cross-section. The base region, which stands vertically, is pictured below. Cross-sections taken perpendicular to the $y$-axis are squares with one side lying on the $xy$-plane.




The variables $x$ and $y$ are given in meters.

**a**. [6 points] Take a slice of soil of thickness $\Delta y$ meters located at $y$ meters above bottom of the hole. Write an expression that approximates the amount of work necessary to move that slice of soil to the top of the hole. The density of the soil is 1600 $\frac{\text{kg}}{\text{m}^3}$. Show all work to receive full credit.

*Solution:*

$$\begin{aligned} W_{slice} &= F_{slice}\ d_{slice} = (\delta_{soil}\ V_{slice}\ g)\ d_{slice} \\ &\approx 1600(2(1+y^2))^2 \Delta y (9.8)(5-y) \\ &= 62720(1+y^2)^2(5-y)\Delta y \end{aligned}$$

**b**. [3 points] How much work does it take to dig the hole? You may use your calculator to answer this question. Include units.

*Solution:*

$$W_{hole} = \int_0^5 62720(1+y^2)^2(5-y)dy = 39{,}984{,}000 \text{ Joules}.$$