Math 116 — Second Midterm

November 16, 2011

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 13 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	12	
2	12	
3	11	
4	11	
5	11	
6	8	
7	13	
8	10	
9	12	
Total	100	

1. [12 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

a. [2 points] The function $y(t) = \cos 3t + B\sin 3t + \frac{1}{9}t$ is a solution of $y'' + 9y = 0$ with $y(0) = 1$.

True | **False**

Solution: $y'' = -9\cos 3t - 9B\sin 3t$ $\quad 9y = 9\cos 3t + 9B\sin 3t + t$
Hence $y'' + 9y = t \neq 0$.

b. [2 points] The value of the integral used to compute the area enclosed by a curve $r = f(\theta)$ given in polar coordinates can be negative if $f(\theta) \leq 0$.

True | **False**

Solution: $f(\theta)^2 \geq 0$, then $A = \int_a^b \frac{1}{2} f(\theta)^2 d\theta \geq 0$.

c. [2 points] If $f(x)$ is a continuous function such that $\int_1^\infty f(x)dx$ diverges, then $\int_1^\infty f(x)^2 dx$ must diverge.

True | **False**

Solution: If $f(x) = \frac{1}{x}$, then $\int_1^\infty \frac{1}{x}dx$ diverges but $\int_1^\infty \frac{1}{x^2}dx$ converges.

d. [2 points] If $P(x)$ is a cumulative distribution function for the probability density function $p(x)$, then $1 + P(x)$ is also a cumulative distribution function for $p(x)$.

True | **False**

Solution: A cumulative distribution function $P(x)$ must satisfy $\lim_{x\to\infty} P(x) = 1$, but $\lim_{x\to\infty} 1 + P(x) = 2 \neq 1$. Hence $1 + P(x)$ can't be a cumulative distribution.

e. [2 points] All solutions to the differential equation $y' = 1 + y^4$ are increasing functions.

True | False

Solution: Since $1 + y^4 > 0$, then $y' > 0$. Then all the solution curves y must be increasing.

f. [2 points] Let $P(t)$ be the cumulative distribution function of a probability density function $p(t)$. If $P(0) = \frac{2}{3}$ then the median of $p(t)$ is negative.

True | False

Solution: If T is the median of $p(t)$, then $P(T) = \frac{1}{2}$. Since $P(0) = \frac{2}{3} > \frac{1}{2}$, then $T < 0$.

2. [12 points] Consider a particle whose trajectory in the xy-plane is given by the parametric curve defined by the equations

$$x(t) = t^4 - 4t^2, \qquad y(t) = t^2 - 2t,$$

for $-3 \leq t \leq 3$. Show all your work to receive full credit.

a. [3 points] Is there any value of t at which the particle ever comes to a stop? Justify.

Solution: No. For the particle to come a stop, its velocity in both the x- and y-direction must be zero. We have that

$$\frac{dx}{dt} = 4t^3 - 8t = 4t(t^2 - 2) = 0$$

at $t = 0, \pm\sqrt{2}$ and

$$\frac{dy}{dt} = 2t - 2 = 0$$

at $t = 1$. Since there are no times at which $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are simultaneously zero, the particle never comes to a stop.

b. [2 points] For what values of t does the path of the particle have a vertical tangent line?

Solution: Vertical tangent lines occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. From the above calculation, this is true at $t = 0, \pm\sqrt{2}$.

c. [3 points] What is the lowest point (x, y) on the curve?

Solution: We want to minimize the value of the y-coordinate over $-3 \leq t \leq 3$. The only critical point for $y(t)$ was found above at $t = 1$. Since $\left.\frac{dy}{dt}\right|_{t=0} = -2 < 0$ and $\left.\frac{dy}{dt}\right|_{t=2} = 2 > 0$, the First Derivative Test tells us that $t = 1$ is a local minimum, and thus a global minimum since it is the only critical point on the given interval. The lowest point on the curve is thus $(x(1), y(1)) = (-3, -1)$.

d. [2 points] At what values of t does the particle pass through the origin?

Solution: We set $x(t) = 0$ and $y(t) = 0$ and solve for t.

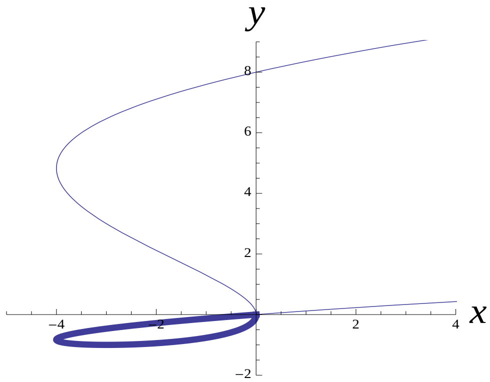
$$x(t) = t^4 - 4t^2 = t^2(t^2 - 4) = t^2(t - 2)(t + 2) = 0$$

gives that $t = -2, 0, 2$, while

$$y(t) = t^2 - 2t = t(t - 2) = 0$$

gives $t = 0, 2$.
Thus, the particle passes through the origin at times $t = 0$ and $t = 2$.

e. [2 points] The graph of the curve traced by these parametric equations is shown below. Find an expression for the length of the closed loop marked in the graph.



Solution: From the given graph and above calculation, we know that the loop is traced out over the time interval $0 \le t \le 2$. The arclength of the loop is given by

$$\int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 \sqrt{(4t^3 - 8t)^2 + (2t-2)^2}\, dt.$$

3. [11 points] In the late 30th century, Mom's Friendly Robot Company is the main global robot manufacturing company. The Bending Unit 22 model is designed to contain a backup unit, effectively rendering it immortal. However, a small percentage of the robots suffer a manufacturing defect, in which the backup unit is malfunctional or not present. The function

$$p(t) = \begin{cases} 0, & \text{if } t < 0 \\ 0.004e^{-t/c}, & \text{if } t \geq 0 \end{cases}$$

gives the probability density for the lifetime of these defective Bending Units 22, where c is a positive constant and t is measured in years since the robots are activated. Show all your work to receive full credit.

a. [2 points] Interpret the quantity $\int_{100}^{140} p(t)\, dt$.

Solution: $\int_{100}^{140} p(t)\, dt$ gives the fraction of defective Bending Units 22 that have a lifespan between 100 and 140 years.

OR

$\int_{100}^{140} p(t)\, dt$ gives the probability that a defective Bending Unit 22 will have a lifetime between 100 and 140 years.

b. [4 points] Find the value of c.

Solution: Since $p(t)$ is a probability density function, we know that $\int_{-\infty}^{\infty} p(t) = 1$. Thus,

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} p(t)\, dt = \int_{-\infty}^{0} p(t)\, dt + \int_{0}^{\infty} p(t)\, dt = 0 + \int_{0}^{\infty} 0.004e^{-t/c}\, dt \\ &= \lim_{b\to\infty} \int_{0}^{b} 0.004e^{-t/c}\, dt = \lim_{b\to\infty} \left[-c \cdot 0.004e^{-t/c}\right]_0^b \\ &= \lim_{b\to\infty} \left(-0.004ce^{-b/c} + 0.004ce^{0}\right) = 0 + 0.004c \\ 1 &= 0.004c \\ c &= 250 \end{aligned}$$

What is the mean (average) lifespan of a defective Bending Unit 22?

c. [5 points]

Solution: Using $c = 250$ from above and that $p(t) = 0$ for $t < 0$,

$$\bar{t} = \int_{-\infty}^{\infty} tp(t)\,dt = \int_{-\infty}^{0} tp(t)\,dt + \int_{0}^{\infty} tp(t)\,dt$$
$$= 0 + \int_{0}^{\infty} 0.004te^{-t/250}\,dt = \lim_{b\to\infty} \int_{0}^{b} 0.004te^{-t/250}\,dt.$$

Integration by parts with $u = t, dv = 0.004e^{-t/250}$(and $du = dt, v = -e^{-t/250}$) gives

$$\bar{t} = \lim_{b\to\infty} \left[-te^{-t/250}\Big|_0^b + \int_0^b e^{-t/250}\,dt \right] = \lim_{b\to\infty} \left[-be^{-b/250} + 0 - 250e^{-t/250}\Big|_0^b \right]$$
$$= \lim_{b\to\infty} \left[-be^{-b/250} - 250e^{-b/250} + 250e^{0} \right] = 0 - 0 + 250$$
$$= 250.$$

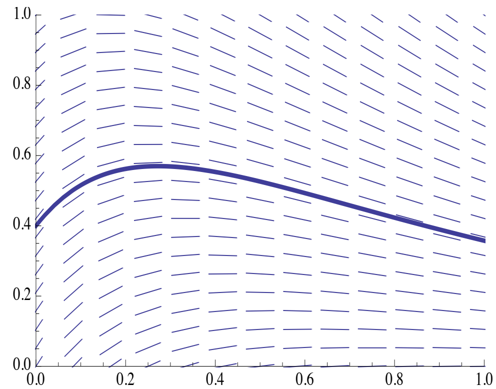
Thus, the mean lifespan of a defective Bending Unit 22 is 250 years.

4. [11 points] A restaurant installs a kitchen ventilation system to control the amount of grease in the air due to cooking. The ventilation system reduces the amount of grease in the air by 90 percent every hour. Let $Q(t)$ be the amount in grams of grease in the air t hours after the ventilation is activated. Then Q satisfies the differential equation

$$\frac{dQ}{dt} = 2e^{-5t} - \frac{9}{10}Q,$$

where $2e^{-5t}$ is the rate at which the kitchen produces grease in grams per hour at time t.

a. [2 points] The slope field of the differential equation is shown below. Suppose that the air in the kitchen initially has 0.4 grams of grease. Sketch the solution curve in the slope field.



b. [7 points] Use Euler's method to estimate the values of the solution curve $Q(t)$ through $(0, 0.4)$ for all values of t given in the table below. Show all your work.

Solution:

t	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$Q(t)$	0.4	.9466	.7885	.5757

$Q_{n+1} = Q_n + (2e^{-5t} - \frac{9}{10}Q_n)\Delta t$ with $\Delta t = \frac{1}{3}$.

$$Q(0) = 0.4 = Q_0$$
$$Q\left(\frac{1}{3}\right) \approx Q_1 = 0.4 + \left(2 - \frac{9}{10}(0.4)\right)\frac{1}{3} \approx .9466$$
$$Q\left(\frac{2}{3}\right) \approx Q_2 = .9466 + \left(2e^{-\frac{5}{3}} - \frac{9}{10}(.9466)\right)\frac{1}{3} \approx .7885$$
$$Q(1) \approx Q_3 = .7885 + \left(2e^{-\frac{10}{3}} - \frac{9}{10}(.7885)\right)\frac{1}{3} \approx .5757.$$

c. [2 points] Does your approximation for $Q(1)$ using Euler's method give an overestimate or an underestimate? Justify.

Solution: The estimate is an overestimate since the solution curves are concave down.

5. [11 points] Determine if the following integrals converge or diverge. If an integral diverges, explain why. If it converges, find the value to which it converges. Mathematical precision is important.

a. [4 points] $\displaystyle\int_{-1}^{2} \frac{1}{\sqrt{2-x}}\,dx$

Solution: The integrand is discontinuous as $x = 2$. Thus, substituting $w = 2 - x, dw = -dx$ we obtain

$$\begin{aligned}
\int_{-1}^{2} \frac{1}{\sqrt{2-x}}\,dx &= \lim_{b\to 2^-} \int_{-1}^{b} (2-x)^{-1/2}\,dx \\
&= \lim_{b\to 2^-} \int_{w(-1)}^{w(b)} w^{-1/2} \cdot -dw \\
&= \lim_{b\to 2^-} -2w^{1/2}\Big|_{w(-1)}^{w(b)} = \lim_{b\to 2^-} -2\sqrt{2-x}\Big|_{-1}^{b} \\
&= \lim_{b\to 2^-} \left(-2\sqrt{2-b} + 2\sqrt{2-(-1)}\right) = 0 + 2\sqrt{3} \\
&= 2\sqrt{3},
\end{aligned}$$

so the integral converges.

b. [4 points] $\displaystyle\int_{10}^{\infty} \frac{5+2\sin(4\theta)}{\theta}\,d\theta$

Solution: Since $-2 \le 2\sin(4\theta) \le 2$, we know that $3 \le 5 + 2\sin(4\theta) \le 7$, and so that in particular

$$\frac{3}{\theta} \le \frac{5+2\sin(4\theta)}{\theta}$$

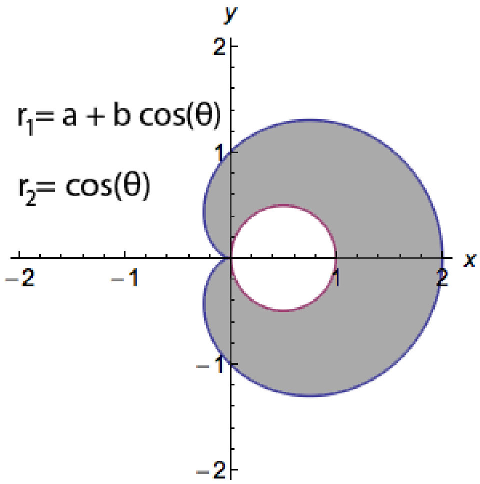
over the interval $[10,\infty)$. Since $\int_{10}^{\infty} \frac{3}{\theta}\,d\theta$ diverges ($p = 1$), we know that $\int_{10}^{\infty} \frac{5+2\sin(4\theta)}{\theta}\,d\theta$ diverges by comparison.

c. [3 points] $\displaystyle\int_{1}^{\infty} \frac{x}{1+x}\,dx$

Solution: Since $\frac{x}{1+x} \to 1$ as $x \to \infty$ (in particular, the integrand does not approach 0), we know that the integral $\int_1^{\infty} \frac{x}{1+x}\,dx$ must diverge.
OR

$\frac{1}{2} \le \frac{x}{x+x} \le \frac{x}{1+x}$ for $x \ge 1$. Hence $\int_1^{\infty} \frac{1}{2}dx \le \int_1^{\infty} \frac{x}{1+x}\,dx$, by the comparison method $\int_1^{\infty} \frac{x}{1+x}\,dx$ diverges.

6. [8 points] Members of the recruitment committee for the Mars University (MU) chapter of the fraternity Epsilon Rho Rho are designing a pledge pin to distribute during Rush Week. The pin takes the shape of a cardioid with a circular hole in it. The cardioid is given by a polar equation of the form $r_1 = a + b\cos(\theta)$, while the circular hole has the polar equation $r_2 = \cos(\theta)$. The pin is pictured below, where the x- and y-axes are measured in inches.



a. [5 points] The committee plans on coating one side of the pin in gold plating, which costs 3 dollars per square inch. Give an expression representing the cost to plate one face of the pin in gold. Your answer may involve integrals and the constants a and b.

Solution: We note that the cardioid $r_1 = a + b\cos(\theta)$ is traced out over $0 \le \theta \le 2\pi$, while the circle $r_2 = \cos(\theta)$ is traced out over $0 \le \theta \le \pi$. The area, in square inches, of the pin is then

$$\frac{1}{2}\int_0^{2\pi} (a + b\cos(\theta))^2 \, d\theta - \frac{1}{2}\int_0^{\pi} (\cos(\theta))^2 \, d\theta,$$

so that the total cost to plate one face of the pin is $\frac{3}{2}\left(\int_0^{2\pi}(a+b\cos(\theta))^2\,d\theta - \int_0^{\pi}\cos^2(\theta)\,d\theta\right)$ dollars. One could also note that the inner circle has radius $\frac{1}{2}$, so that its area is just $\pi\left(\frac{1}{2}\right)^2$, and the cost of the gold plating becomes

$$\frac{3}{2}\int_0^{2\pi} (a + b\cos(\theta))^2 \, d\theta - \frac{3}{4}\pi \text{ dollars.}$$

b. [3 points] Find a and b.

Solution: From the given graph, we see that when $\theta = 0, r_1 = 2$ and when $\theta = \frac{\pi}{2}, r_1 = 1$. This yields the system of equations

$$a + b\cos(0) = a + b = 2, \qquad a + b\cos(\frac{\pi}{2}) = a = 1.$$

Thus $a = 1, b = 1$.
Or
from the given graph, we see that when $\theta = 0, r_1 = 0$ and when $\theta = \frac{\pi}{2}, r_1 = -1$. This yields the system of equations

$$a + b\cos(0) = a + b = 0, \qquad a + b\cos(\frac{\pi}{2}) = a = -1.$$

Thus $a = -1, b = 1$.

7. [13 points] A company designs an air filter for a ship's engine room that reduces the amount of fumes in the air by k percent every hour. The machinery in the engine room produces fumes at a rate of 0.02 kilograms per hour. Let $Q(t)$ be the amount in kilograms of fumes in the room t hours after the engines are activated. Hence Q satisfies

$$\frac{dQ}{dt} = 0.02 - \frac{k}{100}Q.$$

a. [9 points] Find a formula for $Q(t)$. Suppose there are no fumes in the air when the engines are activated.

Solution:

$$\frac{dQ}{0.02 - \frac{k}{100}Q} = dt.$$

$$-\frac{100}{k}\ln\left|0.02 - \frac{k}{100}Q\right| = t + C.$$

$$0.02 - \frac{k}{100}Q = Ae^{-\frac{k}{100}t}$$

$$Q(t) = \frac{100}{k}\left(0.02 - Ae^{-\frac{k}{100}t}\right)$$

$$Q(t) = \frac{2}{k}\left(1 - e^{-\frac{k}{100}t}\right) \qquad \text{using} \qquad Q(0) = 0.$$

b. [2 points] What is the value of $Q(t)$ in the long run?

Solution: $\lim_{t\to\infty} Q(t) = \frac{2}{k}$.

c. [2 points] Air safety regulations require that the *concentration* of fumes in the air not exceed 10^{-4} kilograms per liter at any time. If the volume of air in the engine room is 10^3 liters, for what values of k are the safety regulations met at all times?

Solution: Concentration$=\frac{Q(t)}{\text{Volume}} \le \frac{\frac{2}{k}}{10^3} \le 10^{-4}$.

Hence $k \ge 20$.

8. [10 points] For $\alpha > 0$, consider the family of spirals given by $r = \frac{1}{\theta^\alpha}$ in polar coordinates.

a. [2 points] Write down an integral that gives the length L of a spiral in this family for $1 \leq \theta \leq b$. No credit will be given if you just write down the formula given in part (b).

Solution: Using parametric equations: $x = \frac{1}{\theta^\alpha}\cos\alpha$ and $y = \frac{1}{\theta^\alpha}\sin\alpha$.

$$L = \int_1^b \sqrt{\left(-\frac{\alpha}{\theta^{\alpha+1}}\cos\theta - \frac{1}{\theta^\alpha}\sin\theta\right)^2 + \left(-\frac{\alpha}{\theta^{\alpha+1}}\sin\theta + \frac{1}{\theta^\alpha}\cos\theta\right)^2}\, d\theta$$

or

$$L = \int_1^b \sqrt{\left(\frac{1}{\theta^\alpha}\right)^2 + \left(-\frac{\alpha}{\theta^{\alpha+1}}\right)^2}\, d\theta$$

b. [8 points] It can be shown that the length L of the spiral in part a) may also be written as

$$L = \int_1^b \frac{1}{\theta^\alpha}\sqrt{1 + \frac{\alpha^2}{\theta^2}}\, d\theta.$$

Use this formula for L to find all values of $\alpha > 0$ for which the length of the spiral is infinite for $1 \leq \theta$. For which values of α is the length finite? Justify all your answers using the comparison test.

Solution:

$$\int_1^\infty \frac{1}{\theta^\alpha}\sqrt{1 + \frac{\alpha^2}{\theta^2}}\, d\theta \qquad \text{behaves as} \qquad \int_1^\infty \frac{1}{\theta^\alpha}\, d\theta$$

Since

$$\int_1^\infty \frac{1}{\theta^\alpha}\, d\theta \leq \int_1^\infty \frac{1}{\theta^\alpha}\sqrt{1 + \frac{\alpha^2}{\theta^2}}\, d\theta$$

then arc length is infinite of $\alpha \leq 1$.

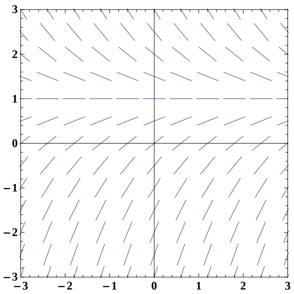
$$\int_1^\infty \frac{1}{\theta^\alpha}\sqrt{1 + \frac{\alpha^2}{\theta^2}}\, d\theta \leq \int_1^\infty \frac{1}{\theta^\alpha}\sqrt{1+\alpha^2}\, d\theta = \sqrt{1+\alpha^2}\int_1^\infty \frac{1}{\theta^\alpha}\, d\theta$$

Hence the arc length is finite of $\alpha > 1$.

9. [12 points] Match each of the following differential equations with its possible slope field. Circle your answers. No justification is required.

(a) $y' = ay$ with $a > 0$. I II III IV

(b) $y' = k(y - a)$ with $a > 0$ and $k < 0$. I II III IV



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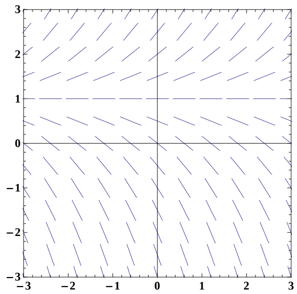
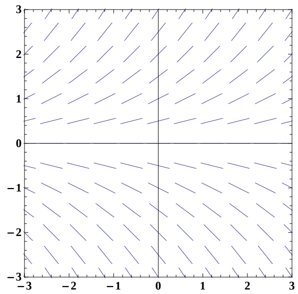


II



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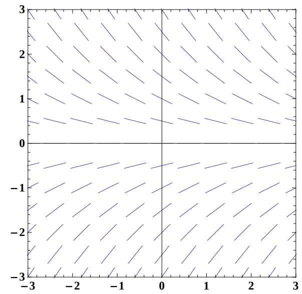
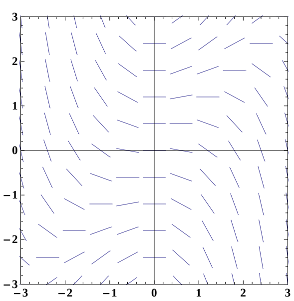


IV

(a) $y' = x(y - ax)$ with $a > 0$. V VI VII VIII

(b) $y' = ay^2 - y$ with $a > 0$. V VI VII VIII



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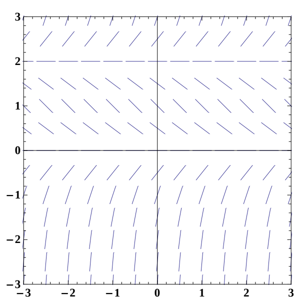

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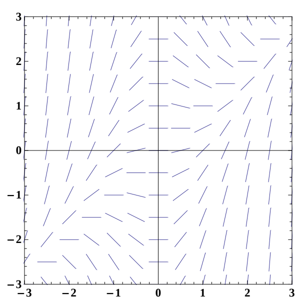
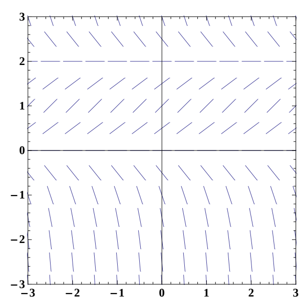


VII



VIII