Math 116 — Final Exam

December 15, 2011

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 14 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	10	
2	12	
3	10	
4	9	
5	10	
6	7	
7	12	
8	12	
9	11	
10	7	
Total	100	

You may find the following expressions useful. And you may not. But you may use them if they prove useful.

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots\cdots \qquad \text{for all values of } x$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots \qquad \text{for } -1 < x < 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\,x^2 + \frac{p(p-1)(p-2)}{3!}\,x^3 + \cdots \qquad \text{for } -1 < x < 1$$

1. [10 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

a. [2 points] Let a_n be a sequence of positive numbers. If $a_n \leq \frac{7^n}{2^{3n}-1}$ for all values of $n \geq 1$, then a_n must converge.

True False

Solution: Since $\lim_{n\to\infty} \frac{7^n}{2^{3n}-1} = \lim_{n\to\infty} \frac{7^n}{8^n-1} = 0$ and $0 < a_n \leq \frac{7^n}{2^{3n}-1}$, $\lim_{n\to\infty} a_n = 0$. Therefore, the sequence a_n converges (to zero).

b. [2 points] The trapezoid rule is guaranteed to give an underestimate of $\int_{-\pi}^{\pi} \cos t dt$.

True **False**

Solution: Since $\cos(t)$ changes concavity over the interval $-\pi \leq t \leq \pi$, the trapezoid rule is guaranteed to give neither an under- nor an overestimate over that interval.

c. [2 points] If the area A under the graph of a positive continuous function $f(x)$ is infinite, then the volume of the solid generated by rotating A around the x-axis could be either infinite or finite depending on the function $f(x)$.

True False

Solution: The area under the graph of $f(x) = \frac{1}{x}$ over $1 \leq x < \infty$ is infinite ($\int_1^\infty \frac{1}{x} dx$ diverges), while the volume of the resulting solid of revolution is given by $\int_1^\infty \pi f(x)^2 dx = \pi \int_1^\infty \frac{1}{x^2} dx$, which converges and so is finite.

On the other hand, the area under the graph of $f(x) = \frac{1}{\sqrt{x}}$ over $1 \leq x < \infty$ is also infinte ($\int_1^\infty \frac{1}{\sqrt{x}} dx$ diverges), and the volume of the resulting solid of revolution is infinite as well ($\int_1^\infty \pi f(x)^2 dx = \pi \int_1^\infty \frac{1}{x} dx$ diverges).

d. [2 points] If $H(x) = \int_0^x f(t)g(t)dt$, then $H'(x) = f'(x)g(x) + f(x)g'(x)$.

True **False**

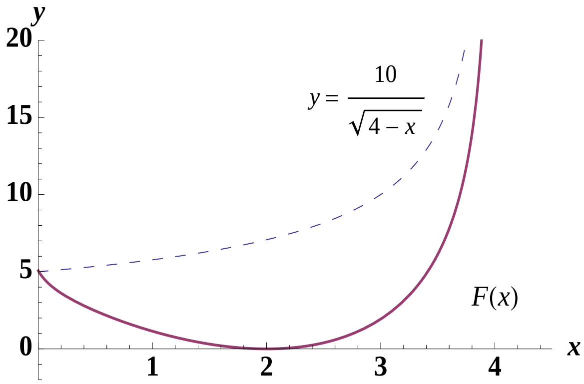
Solution: By the Second Fundamental Theorem of Calculus, $H'(x) = f(x)g(x) \neq f'(x)g(x) + f(x)g'(x)$.

e. [2 points] If $(x(t), y(t))$ gives a parametrization of the unit circle centered at the origin, then $\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = 2\pi$.

True **False**

Solution: $(x(t), y(t))$ need not trace out the unit circle over the interval $0 \leq t \leq 2\pi$. In the case that $(x(t), y(t)) = (\cos(2t), \sin(2t))$, the above integral is equal to 4π, since the unit circle is traced out twice over that interval.

2. [12 points] The graph of $F(x)$ is given below. The function $F(x)$ is defined for $0 \le x < 4$, and its graph is given below. As shown $F(x)$ has a vertical asymptote at $x = 4$. Let $G(x)$ be the antiderivative of $F(x)$ with $G(1) = 1$.



a. [2 points] For what values of x is $G(x)$ increasing?

Solution: Since $G(x)$ is an antiderivative of $F(x)$, $G'(x) = F(x)$. Then $G(x)$ is increasing for $F(x) \ge 0$, which is true for $0 \le x < 4$.

b. [2 points] For what values of x is $G(x)$ concave up?

Solution: Since $G'(x) = F(x), G''(x) = F'(x)$. Therefore, $G(x)$ is concave up where $F'(x) \ge 0$, i.e. when $F(x)$ is increasing. This is true for $2 \le x < 4$.

c. [2 points] Find a formula for $G(x)$ in terms of $F(x)$.

Solution: Since $G(x)$ is the antiderivative of $F(x)$ with $G(1) = 1$, the Construction Theorem (Second Fundamental Theorem of Calculus) gives that $G(x) = 1 + \int_1^x F(t)dt$.

d. [4 points] Is $\int_0^4 \frac{10}{\sqrt{4-x}}dx$ convergent or divergent? If it is convergent, find its exact value.

Solution: The given integral is improper (the integrand is undefined at $x = 4$), so we must use limits to calculate it. Substituting $w = 4 - x, dw = -dx$, we obtain

$$\int_0^4 \frac{10}{\sqrt{4-x}}dx = \lim_{b\to 4^-} \int_0^b \frac{10}{(4-x)^{1/2}}dx = \lim_{b\to 4^-} \int_{w(0)}^{w(b)} \frac{10}{w^{1/2}} \cdot -dw$$

$$= \lim_{b\to 4^-} -20w^{1/2}\Big|_{w(0)}^{w(b)}.$$

In terms of x,

$$\lim_{b\to 4^-} \left(-20\sqrt{4-x}\right)\Big|_0^b = \lim_{b\to 4^-} -20\left(\sqrt{4-b} - \sqrt{4-0}\right) = -20\left(\sqrt{4-4} - \sqrt{4}\right) = 40.$$

So the integral converges.

e. [2 points] Does $\lim\limits_{x\to 4^-} G(x)$ exist? Justify.

Solution: Yes, the given limit exists. This is because $\lim_{x\to 4^-} G(x) = \lim_{x\to 4^-}\left(1+\int_1^x F(t)dt\right) = 1+\int_1^4 F(x)dx$, so the existence of the given limit depends on whether or not the integral $\int_1^4 F(x)dx$ converges. We are given that $F(x) \leq \frac{10}{\sqrt{4-x}}$ over $1 \leq x < 4$, then

$$\int_1^4 F(x)dx \leq \int_1^4 \frac{10}{\sqrt{4-x}}dx$$

and since $\int_1^4 \frac{10}{\sqrt{4-x}}dx$ converges (by a similar calculation to part **d.**), $\int_1^4 F(x)dx$ converges by the comparison method.

3. [10 points] In order to fuel a late-night study session at the UGLi, you pull out a can of Bolt Kola, a highly caffeinated soft drink. This particular brand of pop comes in a cylindrical, aluminum can with a removable top. You want to know how much force is exerted on the sides of the can by the drink. You pull out your trusty ruler and find the base of the can has a radius of 3.5 centimeters (0.035 meters) and a height of 16 centimeters (0.16 meters). A quick Google search informs you that the density of Bolt Kola is 1030 kg/m^3.

a. [2 points] Calculate the force exerted by the drink on the bottom of the can.

Solution: Since the bottom of the can is at a constant depth of 0.16 meters, pressure is constant and so the force exerted on it is given by Force = Pressure $\times$ Area. Since pressure is given by $P =$ density $\cdot\, g \,\cdot$ depth, we have

$$F_{\text{bottom}} = 1030 \cdot 9.8 \cdot 0.16 \cdot \pi\,(0.035)^2 \approx 6.2154 \text{ Newtons.}$$

b. [5 points] Write an expression giving the force exerted by the drink on a slice of the cylindrical wall of the can h meters above the base and of thickness Δh.

Solution: Since $h = 0$ at the bottom of the can, the depth of the slice is $(0.16 - h)$. The pressure at this depth is then $P_{sl} = 1030 \cdot 9.8(0.16 - h) = 10094(0.16 - h)$. The area of the slice is $A_{sl} = 2\pi r \Delta h = 2\pi(0.035)\Delta h = 0.07\pi\Delta h$. Thus, the force exerted on the slice is

$$F_{slice} = P_{slice} \times A_{slice} = 10094(0.16 - h) \times 0.07\pi\Delta h = 706.58\pi(0.16 - h)\Delta h.$$

c. [3 points] Calculate the total force exerted by the drink on the sides of the can (with the top removed). Show all work to receive full credit.

Solution: The force exerted on the side of the can is

$$\int_0^{0.16} 706.58\pi(0.16 - h)dh = 706.58\pi\left(0.16h - \frac{h^2}{2}\right)\Big|_0^{0.16} \approx 28.4133 \text{ Newtons.}$$

Thus the *total* force exerted on the can (without the top) is approximately

$$28.4133 + F_{\text{bottom}} \approx 28.4133 + 6.2154 = 34.6287 \text{ Newtons.}$$

4. [9 points] Ramon starts depositing \$10,000 each year at his 25th birthday into a retirement account and continues until his 45th birthday. After this point, he does not touch the account until he is 65. The retirement account accrues interest at a rate of 3% compounded annually.

a. [3 points] Let R_n be the amount of money *in thousands* of dollars in Ramon's retirement account after n years from his initial deposit. Find an expression for R_0, R_1 and R_2.

Solution:
$R_0 = 10$, since Ramon deposits \$10,000 initially.
$R_1 = 10 + 10(1.03)$, since Ramon deposits another \$10,000 and the previous year's deposit accrues interest.
$R_2 = 10 + [10 + 10(1.03)]\,(1.03) = 10 + 10(1.03) + 10(1.03)^2$, since all of R_1 accrues interest.

b. [3 points] Find a closed form expression (an expression that does not involve a long summation) for how much money Ramon has in his retirement account at his 45th birthday.

Solution: From the calculations in part (a), we can see that

$$R_n = 10 + 10(1.03) + 10(1.03)^2 + \ldots + 10(1.03)^n.$$

Ramon's 45th birthday corresponds to $n = 20$, so

$$R_{20} = 10 + 10(1.03) + 10(1.03)^2 + \ldots + 10(1.03)^{20} = \sum_{k=0}^{20} 10(1.03)^k.$$

As this is a finite geometric series with 21 terms, a closed form expression is

$$R_{20} = \frac{10(1 - 1.03^{21})}{1 - 1.03},$$

where the amount is given in thousands of dollars.

c. [3 points] Find a closed form expression for how much money Ramon has in his retirement account when he is 65 years old. Compute its value.

Solution: Since Ramon stops depositing money after his 45th birthday, his account is just accumulating interest (at an annual rate of 3%) for the next 20 years. Thus, when he is 65 years old, his account balance is

$$R_{20}\,(1.03)^{20} = \frac{10(1 - 1.03^{21})}{1 - 1.03}\,(1.03)^{20} \approx 517.92923$$

in thousands of dollars (\$517,929.23).

5. [10 points] When a voltage V in volts is applied to a series circuit consisting of a resistor with resistance R in ohms and an inductor with inductance L, the current $I(t)$ at time t is given by

$$I(t) = \frac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \qquad \text{where } V, R, \text{ and } L \text{ are constants.}$$

a. [2 points] Show that $I(t)$ satisfies

$$\frac{dI}{dt} = \frac{V}{L}\left(1 - \frac{R}{V}I\right).$$

Solution:

$$\frac{dI}{dt} = \frac{V}{R}\left(-e^{-Rt/L} \cdot -\frac{R}{L}\right) = \frac{V}{L}e^{-Rt/L} = \frac{V}{L}\left(1 - (1 - e^{-Rt/L})\right)$$
$$= \frac{V}{L}\left(1 - \frac{R}{V}I\right)$$

b. [6 points] Find a Taylor series for $I(t)$ about $t = 0$. Write the first three nonzero terms and a general term of the Taylor series.

Solution: Instead of taking derivatives of $I(t)$ at $t = 0$, we can use the given expansion for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$ about 0. Here, $x = \frac{-Rt}{L}$.

$$\begin{aligned}
I(t) &= \frac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \\
&= \frac{V}{R}\left[1 - \left(1 + (\frac{-Rt}{L}) + \frac{(-\frac{Rt}{L})^2}{2!} + \frac{(-\frac{Rt}{L})^3}{3!} + \ldots + \frac{(-\frac{Rt}{L})^n}{n!} + \ldots\right)\right] \\
&= \frac{V}{R}\left[1 - \left(1 - \frac{R}{L}t + \frac{R^2}{2!L^2}t^2 - \frac{R^3}{3!L^3}t^3 + \ldots + \frac{(-1)^n R^n}{n!L^n}t^n + \ldots\right)\right] \\
&= \frac{V}{R}\left(\frac{R}{L}t - \frac{R^2}{2!L^2}t^2 + \frac{R^3}{3!L^3}t^3 - \ldots + \frac{(-1)^{n+1}R^n}{n!L^n}t^n + \ldots\right) \\
&= \frac{V}{L}t - \frac{VR}{2!L^2}t^2 + \frac{VR^2}{3!L^3}t^3 - \ldots + \frac{(-1)^{n+1}VR^{n-1}}{n!L^n}t^n + \ldots
\end{aligned}$$

c. [2 points] Use the Taylor series to compute

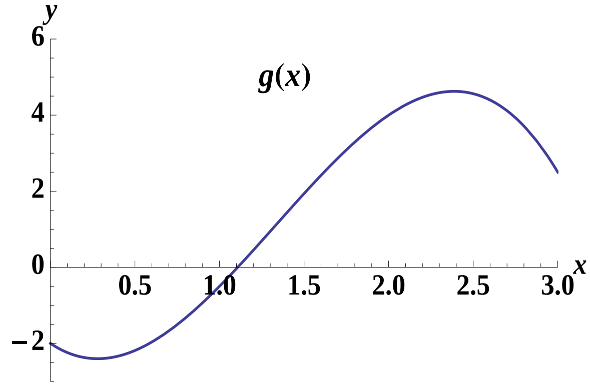
$$\lim_{t\to 0}\frac{I(t)}{t}.$$

Solution:

$$\begin{aligned}\lim_{t\to 0}\frac{I(t)}{t} &= \lim_{t\to 0}\frac{1}{t}\left(\frac{V}{L}t-\frac{VR}{2!L^2}t^2+\frac{VR^2}{3!L^3}t^3-\ldots+\frac{(-1)^{n+1}VR^{n-1}}{n!L^n}t^n+\ldots\right)\\ &= \lim_{t\to 0}\left(\frac{V}{L}-\frac{VR}{2!L^2}t^1+\frac{VR^2}{3!L^3}t^2-\ldots+\frac{(-1)^{n+1}VR^{n-1}}{n!L^n}t^{n-1}+\ldots\right)\\ &= \frac{V}{L},\end{aligned}$$

since all of the terms involving t evaluate to 0.

6. [7 points] Let $f(x) = \frac{1}{3x-2}$ and $g(x)$ be the function whose graph is shown below.



a. [3 points] Let $P_2(x) = a + b(x-2) + c(x-2)^2$ be the second order Taylor polynomial approximating $g(x)$ for x near 2. What can you say about the signs of the coefficients a, b and c?

Solution: Since $P_2(x) = g(2) + g'(2)(x-2) + \frac{g''(2)}{2}(x-2)^2$, then $a = g(2)$, $b = g'(2)$ and $c = \frac{g''(2)}{2}$. At $x = 2$ the function $g(x)$ is positive ($g(2) > 0$), (increasing $g'(2) > 0$) and concave down ($g''(2) < 0$). Hence $a > 0$, $b > 0$ and $c < 0$.

b. [4 points] Find the second order Taylor polynomial approximating $f(x)$ for x near -1.

Solution:

$$f(x) = \frac{1}{3x-2} \qquad f(-1) = -\frac{1}{5}$$
$$f'(x) = \frac{-3}{(3x-2)^2} \qquad f'(-1) = -\frac{3}{25}$$
$$f''(x) = \frac{18}{(3x-2)^3} \qquad f''(-1) = -\frac{18}{125}.$$

Hence $P_2(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2}(x+1)^2$

$$P_2(x) = -\frac{1}{5} - \frac{3}{25}(x+1) - \frac{9}{125}(x+1)^2.$$

7. [12 points] Determine whether the following series converge or diverge (circle your answer). Be sure to mention which tests you used to justify your answer. If you use the comparison test or limit comparison test, write an appropriate comparison function.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{1+2\sqrt{n}}$

Solution: Since

$$\lim_{n\to\infty}\frac{\sqrt{n}}{1+2\sqrt{n}}=\lim_{n\to\infty}\frac{\sqrt{n}}{2\sqrt{n}}=\frac{1}{2},$$

then the sequence $a_n=(-1)^n\frac{\sqrt{n}}{1+2\sqrt{n}}$ does not converge to zero (it oscillates closer to $\frac{1}{2}$ and $-\frac{1}{2}$. Since the terms a_n does not converge to 0, then the series $\sum_{n=1}^{\infty}a_n$ diverges.

b. [4 points] $\displaystyle\sum_{n=1}^{\infty}ne^{-n^2}$

Solution: Let $f(x)=xe^{-x^2}$. The function $f(x)>0$ and $f'(x)=e^{-x^2}(1-2x^2)<0$ for $x\geq 1$. Hence by the Integral test

$$\sum_{n=1}^{\infty}ne^{-n^2}\text{ behaves as }\int_1^{\infty}xe^{-x^2}dx=\lim_{b\to\infty}\int_1^b xe^{-x^2}dx=\lim_{b\to\infty}-\frac{1}{2}e^{-x^2}|_1^b=\frac{1}{2e}$$

the series converges.

c. [5 points] $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n^2)}{n^2}$

Solution: This series has positive and negative terms. Since

$$\left|\frac{\cos(n^2)}{n^2}\right|\leq\frac{1}{n^2},$$

then the series of the absolute values satisfies

$$\sum_{n=1}^{\infty}\left|\frac{\cos(n^2)}{n^2}\right|\leq\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

The series on the right converges by the p series test with $p=2$, hence the series of absolute values converges. Since the series of absolute values converges, then $\sum_{n=1}^{\infty}\frac{\cos(n^2)}{n^2}$ converges.

8. [12 points] Let

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(x+1)^{2n}$$

a. [3 points] At $x = -3$, does the series converge absolutely, conditionally or diverge?

Solution: At $x = -3$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(x+1)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(-2)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

The sequence $a_n = \frac{1}{n+1}$ is decreasing and converges to 0. By the Alternating series test, the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges. The convergence is not absolute since

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n+1} \right| = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

which diverges by p series test with $p = 1$. Hence the series converges conditionally at $x = -3$.

b. [2 points] Using just your answer in (a), state the possible values for the radius of convergence R could be. Justify.

Solution:
Solution 1: Since the center of the series is $a = -1$ and the series converges at $x = -3$, then $R \geq 2$.
Solution 2: Since power series converges absolutely inside its interval of convergence and at $x = -3$, the series converges conditionally, then $R = 2$.

c. [7 points] Find the interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(x+1)^{2n}$$

Solution:
Solution 1: Using Ratio Test

$$\lim_{n\to\infty} \frac{\left| \frac{(-1)^{n+1}}{4^{n+1}(n+2)}(x+1)^{2n+2} \right|}{\left| \frac{(-1)^n}{4^n(n+1)}(x+1)^{2n} \right|} = \frac{|x+1|^2}{4} \lim_{n\to\infty} \frac{n+1}{n+2} = \frac{|x+1|^2}{4} = L.$$

Then the series converges if $\frac{|x+1|^2}{4} < 1$ or if $-3 < x < 1$. We already know from (a) that at $x = -3$ the series converges. If $x = 1$, then the series become $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ which converges by (a). Hence the interval of convergence is $[-3, -1]$.

Solution 2: Since the radius of convergence is equal to 2, then we only need to check the other endpoint of the interval of convergence $x = 1$. If $x = 1$, then the series become $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ which converges by (a). Hence the interval of convergence is $[-3, -1]$.

9. [11 points] Philip J. Frye has a bank account with Big Apple Bank that compounds interest at a continuous annual rate of 1%. His account has a balance of \$300 at midnight of January 1, 2000, when Frye is cryogenically frozen for 1000 years. The entire time he is frozen, his account accumulates interest. Include units in your answers where appropriate.

a. [2 points] Write a differential equation that models Frye's account balance $M(t)$, where M is given in dollars and t is the number of years since January 1, 2000. List any initial conditions.

Solution:

$$\frac{dM}{dt} = 0.01M \qquad M(0) = 300.$$

b. [4 points] Solve your differential equation from (a) to find the balance in Frye's account after he is awoken in the year 3000. Show all your work.

Solution:

$$\frac{dM}{dt} = 0.01M \qquad \frac{dM}{M} = 0.01dt \qquad \ln|M| = 0.01t + C$$

$$M(t) = Be^{0.01t}$$

$$M(0) = 300 \qquad 300 = B \qquad M(t) = 300e^{0.01t}.$$

c. [2 points] Suppose that Big Apple Bank charges an annual fee of \$5 to maintain the account, withdrawn continuously over the course of the year. Write a new differential equation for $M(t)$, the balance in Frye's bank account.

Solution:

$$\frac{dM}{dt} = 0.01M - 5.$$

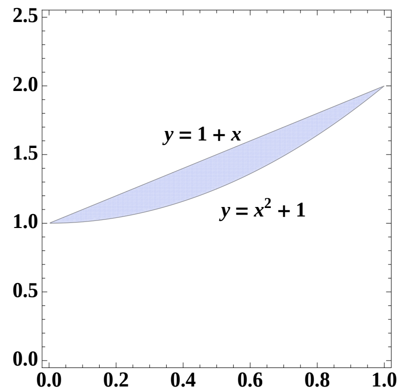
d. [3 points] How large must the initial deposit in Frye's account be at Big Apple Bank in order for the account to be profitable for him? Justify your answer mathematically.

Solution: The differential equation has an equilibrium solution at $M_{eq} = \frac{5}{0.01} = 500$. The equilibrium solution is unstable since

$$\left.\frac{dM}{dt}\right|_{M=400} = .01(400) - 5 = -1 < 0 \qquad \text{and} \quad \left.\frac{dM}{dt}\right|_{M=600} = .01(600) - 5 = 1 > 0$$

Hence the initial deposit $M(0)$ has to be larger than $M_{eq} = 500$ dollars in order to be profitable ($M(0) > 500$).

10. [7 points] A metal thin plate has density $\delta(x) = 1 + x$ kg per square meter. The shape of the plate is bounded by the curves $y = 1 + x$ and $y = 1 + x^2$ for $0 \leq x \leq 1$ as shown below.



a. [3 points] Find the exact value of the mass of the plate. Show all your work.

Solution:

$$m = \int_0^1 (1 + x - 1 - x^2)(1 + x)dx = \int_0^1 (x - x^2)(1 + x)dx = \int_0^1 x - x^3 dx$$

$$= \frac{x}{2} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \text{ kg.}$$

b. [4 points] Let $(\bar{x}, \bar{y})$ be its center of mass. Write a formula for $\bar{x}$ and evaluate your formula to find the exact value of $\bar{x}$. Show all your work.

Solution:

$$\bar{x} = \frac{\int_0^1 x(x - x^3)dx}{\frac{1}{4}} = 4\int_0^1 x^2 - x^4 dx = 4\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 = 4\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8}{15} \text{ meters .}$$