Math 116 — Second Midterm

November 14 , 2012

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	8	
2	14	
3	15	
4	11	
5	12	
6	13	
7	13	
8	14	
Total	100	

1. [8 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

a. [2 points] The function $y(t) = \cos(4t)$ is a solution to the differential equation $y'' + 16y = 0$.

$\boxed{\text{True}}$ False

Solution: $y' = -4\sin(4t)$ and $y'' = -16\cos(4t)$.
Hence $y'' + 16y = -16\cos(4t) + 16\cos(4t) = 0$.

b. [2 points] $\int_1^2 \tan x \, dx$ is an improper integral.

$\boxed{\text{True}}$ False

Solution: The function $y = \tan(x)$ has a vertical asymptote at $x = \frac{\pi}{2} \approx 1.5708$ which lies inside the interval $[1, 2]$. Hence $\int_1^2 \tan x \, dx$ is improper.

c. [2 points] If $r = f(\theta)$ is a function in polar coordinates with $f''(\theta) > 0$, then its graph in the x-y plane is concave up.

True $\boxed{\text{False}}$

Solution: Consider $f(\theta) = e^{-\theta}$, then $f''(\theta) = e^{-\theta} > 0$ for all θ, but its graph is not concave up for all θ.

d. [2 points] The median of the probability density function

$$p(x) = \begin{cases} \frac{1}{x^2} & x \geq 1. \\ 0 & x < 1. \end{cases}$$

is equal to 2.

$\boxed{\text{True}}$ False

Solution: Since

$$\int_{-\infty}^{2} p(t)dt = \int_1^2 \frac{1}{x^2} dx = -\frac{1}{x}\bigg|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}.$$

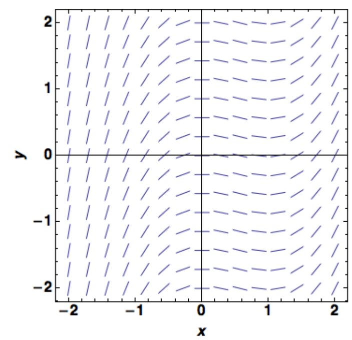
Hence 2 is the median of this distribution.

2. [14 points]

a. [10 points] Consider the following differential equations:

A. $y' = x(y-2)$ B. $y' = x(x-1)$ C. $y' = (x-y)y$ D. $y' = (2-y)(y+1)^2$

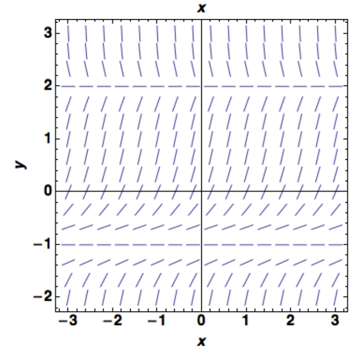
Each of the following slope fields belongs to one of the differential equations listed above. Indicate which differential equation on the given line. Find the equation of the equilibrium solutions and their stability. If a slope field has no equilibrium solutions, write none.



Differential equation: **B**

Equilibrium solutions and stability:

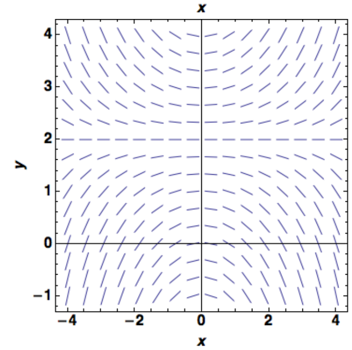
None



Differential equation: **D**

Equilibrium solutions and stability:

- $y = -1$ unstable (or semistable).
- $y = 2$ stable.



Differential equation: **A**

Equilibrium solutions and stability:

$y = 2$ unstable.

b. [4 points] Find the regions in the x-y plane where the solution curves to the differential equation $y' = (y - x^2)y$ are increasing.

Solution: $y' > 0$ if:

- $y > x^2$, or
- $y < 0$.

3. [15 points] A model for cell growth states that the volume $V(t)$ (in mm^3) of a cell at time t (in days) satisfies the differential equation

$$\frac{dV}{dt} = 2e^{-t}V.$$

a. [2 points] Find the equilibrium solutions of this equation.

Solution: $V = 0$.

b. [8 points] Solve the differential equation. The initial volume of the cell is V_0 mm^3. Your answer should contain V_0.

Solution:

$$\begin{aligned} \frac{dV}{dt} &= 2e^{-t}V \\ \frac{dV}{V} &= 2e^{-t}dt \\ \ln|V| &= -2e^{-t} + C \\ V &= Be^{-2e^{-t}}. \\ V_0 &= Be^{-2} \\ B &= V_0e^2 \\ V &= V_0e^2e^{-2e^{-t}} = V_0e^{2-2e^{-t}}. \end{aligned}$$

c. [3 points] How long does it take a cell to double its initial size?

Solution:

$$\begin{aligned} 2V_0 &= V_0e^{2-2e^{-t}} \\ 2 &= e^{2-2e^{-t}} \\ \ln 2 &= 2 - 2e^{-t} \\ 2e^{-t} &= 2 - \ln 2 \\ e^{-t} &= \frac{2 - \ln 2}{2} \\ t &= -\ln\left(\frac{2 - \ln 2}{2}\right). \end{aligned}$$

d. [2 points] What happens to the value of the volume of the cell in the long run?

Solution: $\lim_{t\to\infty} V(t) = \lim_{t\to\infty} V_0e^{2-2e^{-t}} = V_0e^2$. Hence the volume of the cell $V(t)$ approaches the value V_0e^2.

4. [11 points] The function $P(t)$ models the number of bees (in thousands) in a colony at time t (in years). Suppose the function $P(t)$ satisfies the differential equation

$$\frac{dP}{dt} = 2(1 - 2\sin t)P.$$

The colony initially has 500 bees.

a. [6 points] Use Euler's method, with three steps, to find the approximate number of bees (in thousands) in the farm after one year. Fill in the table with the appropriate values of t and your approximations.

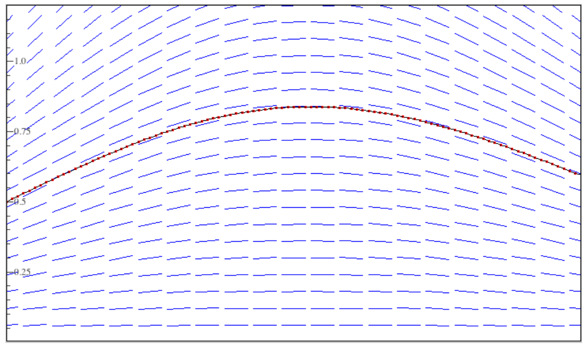
t (in years)	0			1
$P(t)$ (in thousands)				

Solution:

t	P	$\Delta P = 2(1 - 2\sin t)P\Delta t$
0	0.5	$2(1 - 2\sin 0)(0.5)(\frac{1}{3}) = \frac{1}{3}$
$\frac{1}{3}$	0.833	$2(1 - 2\sin\frac{1}{3})(0.833)(\frac{1}{3}) = .191$
$\frac{2}{3}$	1.025	$2(1 - 2\sin\frac{2}{3})(1.025)(\frac{1}{3}) = -0.161$
1	0.863	

b. [1 point] The slope field of the differential equation $\frac{dP}{dt} = 2(1 - 2\sin t)P$ is shown below. Use it to sketch the graph of $P(t)$, the number of bees (in thousands) in the colony after t years.

Solution:

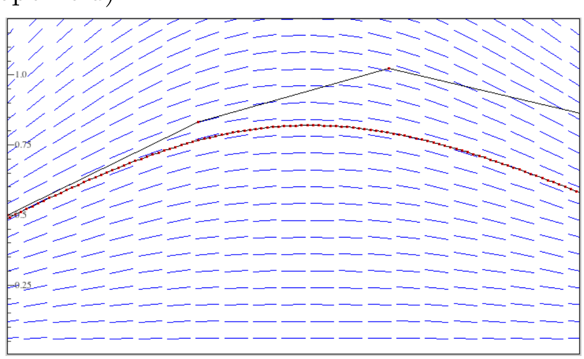


c. [2 points] Use the differential equation $\frac{dP}{dt} = 2(1 - 2\sin t)P$ to find the exact value of t during the first year at which the number of bees in the colony has a maximum.

Solution: In order to get $\frac{dP}{dt} = 0$ we need either $2(1 - 2\sin t) = 0$ or $P = 0$. Since the number of bees $P(t)$ is never zero in the first year as seen in the slope field above, then at the maximum $2(1 - 2\sin t) = 0$. This occurs when $\sin t = \frac{1}{2}$. During the first year it is at $t = \frac{\pi}{6} \approx .523$ years.

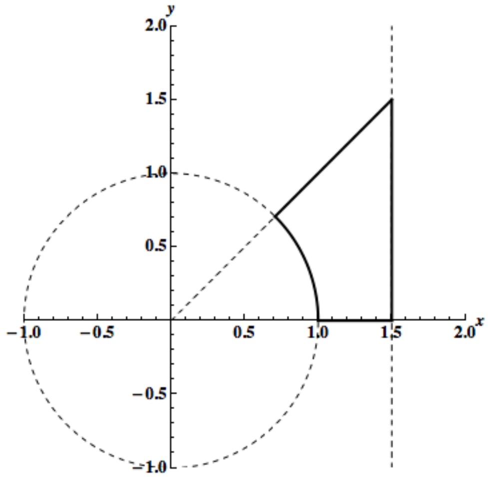
d. [2 points] Does the approximation of $P(1)$ obtained with Euler's method in (a) guarantee an underestimate, an overestimate or neither? Justify without solving the differential equation.

Solution: Euler's method yields an overestimate for $P(1)$ since the function $P(t)$ is concave down (see slope field).



5. [12 points] Solve each of the following problems.

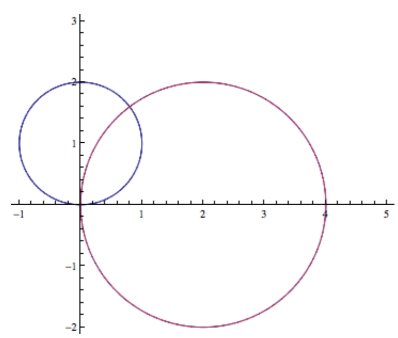
a. [4 points] Give inequalities for r and θ that describe the region shown below in polar coordinates. The region is bounded by the circle $x^2 + y^2 = 1$, the line $y = x$, the x-axis and the vertical line $x = 1.5$.



Solution: The line $y = x$ is the polar line $\theta = \frac{\pi}{4}$, so the limits for θ are $0 \le \theta \le \frac{\pi}{4}$. The values of r range from $r = 1$ on the circle to the line $r\cos\theta = 1.5$, or $r = \frac{1.5}{\cos\theta}$. So the region is

$$\begin{cases} 0 \le \theta \le \frac{\pi}{4} \\ 1 \le r \le \frac{1.5}{\cos\theta} \end{cases}$$

b. [8 points] The functions in polar coordinates $r = 2\sin\theta$ and $r = 4\cos\theta$ represent the circles shown below



Let A be the area of the intersection of these circles. Find an expression involving definite integrals in polar coordinates that computes the value of A. You do not need to evaluate the integrals.

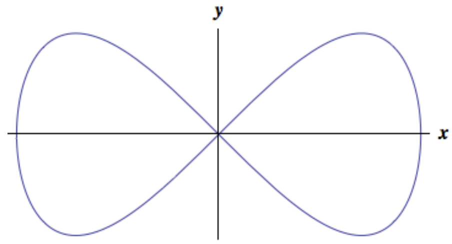
Solution: The curves intersect where $2\sin\theta = 4\cos\theta$, or $\tan\theta = 2$. On the interval $0 \le \theta \le \arctan(2)$, $r = 2\sin\theta$ is the outside curve. On the interval $\arctan(2) \le \theta \le \frac{\pi}{2}$, $r = 4\cos\theta$ is the outside curve. The area is

$$A = \frac{1}{2}\int_0^{\arctan(2)} 4\sin^2\theta d\theta + \frac{1}{2}\int_{\arctan(2)}^{\frac{\pi}{2}} 16\cos^2\theta d\theta.$$

6. [13 points] A particle moves along the path given by the parametric equations

$$x(t) = a\cos t \qquad y(t) = \sin 2t \qquad \text{for } 0 \le t \le 2\pi.$$

where a is a positive constant. The graph of the particle's path in the x-y plane is shown below. In the questions below, show all your work to receive full credit.



a. [2 points] At which values of $0 \le t \le 2\pi$, does the particle pass through the origin?

Solution: $0 = x(t) = a\cos t$: $t = \frac{\pi}{2}, \frac{3\pi}{2}$.
$0 = y(t) = \sin 2t$: $2t = 0, \pi, 2\pi, 3\pi, 4\pi \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. Particle passes through origin: $t = \frac{\pi}{2}, \frac{3\pi}{2}$.

b. [5 points] For what values of a are the two tangent lines to the curve at the origin perpendicular? Hint: Two lines are perpendicular if the product of their slopes is equal to -1.

Solution: $x'(t) = -a\sin t$, $y'(t) = 2\cos 2t$.

$t = \frac{\pi}{2}$	$t = \frac{3\pi}{2}$
$\frac{dx}{dt} = -a$	$\frac{dx}{dt} = a$
$\frac{dy}{dt} = -2$	$\frac{dy}{dt} = -2$
$\frac{dy}{dx} = \frac{2}{a}$	$\frac{dy}{dx} = -\frac{2}{a}$

$$-1 = \frac{2}{a}\left(-\frac{2}{a}\right) = -\frac{4}{a^2} \Rightarrow a = 2.$$

c. [4 points] At what values of $0 \le t \le 2\pi$, does the curve have horizontal tangents?

Solution:

$$0 = y'(t) = 2\cos 2t \Rightarrow 2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots, \Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

d. [2 points] Find an expression that computes the length of the curve.

Solution:

$$\int_0^{2\pi} \sqrt{a^2\sin^2 t + 4\cos^2 2t}\,dt.$$

7. [13 points] Consider the following improper integrals. Show all your work to receive full credit.

a. [5 points] Determine the convergence or divergence of the following improper integral. If the integral converges, compute its value.

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx$$

Solution: The integral is improper at $x = 0$ since $\sin 0 = 0$. Changing to a limit of proper integrals and using the substitution $u = \sin x$:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx &= \lim_{a \to 0^+} \int_a^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx \\ &= \lim_{a \to 0^+} \int_{\sin a}^{1} u^{-1/2} du \\ &= \lim_{a \to 0^+} 2u^{1/2}\Big|_{\sin a}^{1} \\ &= \lim_{a \to 0^+} 2 - 2\sqrt{\sin a} \\ &= 2.\end{aligned}$$

Determine the convergence or divergence of the following improper integrals. Circle your answers.

b. [4 points] $\displaystyle\int_2^{\infty} \frac{5 - 3\sin(2x)}{x^2} dx$ $\boxed{\text{Converges}}$ Diverges

Solution: Since $0 \le 5 - 3\sin(2x) \le 8$,

$$\int_2^{\infty} \frac{5 - 3\sin(2x)}{x^2} dx \le 8 \int_2^{\infty} \frac{1}{x^2} dx,$$

which converges by the p-test with $p = 2$.

c. [4 points] $\displaystyle\int_1^{\infty} \frac{1}{x}\sqrt{a^2 + \frac{1}{\sqrt{x}}}\, dx$, where a is a positive constant.

Converges $\boxed{\text{Diverges}}$

Solution: Since $a^2 + \frac{1}{\sqrt{x}} \ge a^2$ for $x > 0$,

$$\frac{1}{x}\sqrt{a^2 + \frac{1}{\sqrt{x}}} \ge \frac{a}{x},$$

and so

$$\int_1^{\infty} \frac{1}{x}\sqrt{a^2 + \frac{1}{\sqrt{x}}} dx \ge a \int_1^{\infty} \frac{1}{x} dx,$$

which diverges by the p-test, with $p = 1$.

8. [14 points] A coffee shop offers only one hour of free internet access to all its customers. The time t in hours a customer uses the internet at the coffee shop has a probability density function

$$p(t) = \begin{cases} at\sqrt{1-t^2} & 0 \le t \le 1. \\ 0 & \text{otherwise.} \end{cases}$$

where a is a constant.

a. [4 points] For what value of a is $p(t)$ a probability density function? Find its value without using your calculator.

Solution:

$$1 = \int_0^1 at\sqrt{1-t^2}dt = -\frac{a}{2}\int_1^0 \sqrt{u}du = -\frac{a}{2}\,\frac{2}{3}u^{3/2}\Big|_1^0 = -\frac{a}{2}\left(-\frac{2}{3}\right) = \frac{a}{3}.$$

So, $a = 3$.

b. [4 points] Find the cumulative distribution function $P(t)$ of $p(t)$. Make sure to indicate the value of $P(t)$ for all values of $-\infty < t < \infty$. Your final answer should not contain any integrals.

Solution: $P(t) = \int_{-\infty}^{t} p(x)dx$, so if $t \le 0$ then $P(t) = 0$, if $t \ge 1$ then $P(t) = 1$. If $0 < t < 1$,

$$P(t) = \int_0^t 3x\sqrt{1-x^2}dx = -\frac{3}{2}\int_1^{1-t^2} \sqrt{u}du = -\,u^{3/2}\Big|_1^{1-t^2} = 1-(1-t^2)^{3/2}.$$

c. [3 points] Find the the probability that a customer is still using the internet after 40 minutes (without using your calculator).

Solution: The probability that a customer users the internet for 40 minutes or less is $P(40/60) = P(2/3)$. So the probability of using the internet for more than 40 minutes is

$$1 - P(2/3) = 1 - \left(1 - (1 - (2/3)^2)^{3/2}\right) = \left(1 - \frac{4}{9}\right)^{3/2} = \frac{\sqrt{125}}{27}.$$

d. [3 points] Find an expression for the mean of this distribution. Use your calculator to compute its value.

Solution:

$$\int_0^1 3t^2\sqrt{1-t^2}dt \approx 0.589 \text{ hours.}$$