Math 116 — Final Exam

December 14, 2012

Name: EXAM SOLUTIONS

Instructor: ____________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	12	
2	9	
3	12	
4	11	
5	9	
6	12	
7	14	
8	10	
9	11	
Total	100	

You may find the following expressions useful.

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

1. [12 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

a. [2 points] In polar coordinates, $(r_1, \theta_1) = (2, \frac{\pi}{5})$ and $(r_2, \theta_2) = (-2, -\frac{4\pi}{5})$ represent the same point.

True False

b. [2 points] If a particle moves according to the parametric equations $x(t) = t^3 + t^2$ and $y(t) = t^4$, then the particle has speed zero at the origin.

True False

Solution: The speed of the particle is given by $v(t) = \sqrt{(x')^2 + (y')^2}$. In this case $v(t) = \sqrt{(3t^2 + 2t)^2 + (4t^3)^2}$. The particle is at the origin when $t = 0$ and $v(0) = 0$.

c. [2 points] The Taylor series for $f(x) = \sqrt{1 + 2x}$ centered about $x = 0$ converges for $-1 < x < 1$.

True **False**

Solution: The Taylor series for $f(x) = \sqrt{1 + 2x}$ centered about $x = 0$ converges for $-\frac{1}{2} < x < \frac{1}{2}$.

d. [2 points] If $P(t)$ is a cumulative distribution function, then the sequence $x_n = P(n)$ converges.

True False

Solution: If $P(t)$ is a cumulative distribution function, then $\lim_{t\to\infty} P(t) = 1$, hence $\lim_{n\to\infty} P(n) = 1$.

e. [2 points] The sequence $a_n = \int_{\frac{1}{n}}^{1} \frac{1}{x^3}\, dx$ converges.

True **False**

Solution: $\lim_{t\to\infty} a_n = \lim_{n\to\infty} \int_{\frac{1}{n}}^{1} \frac{1}{x^3}\, dx = \int_0^1 \frac{1}{x^3}\, dx$ which diverges by p-series test with $p = 3 > 1$.

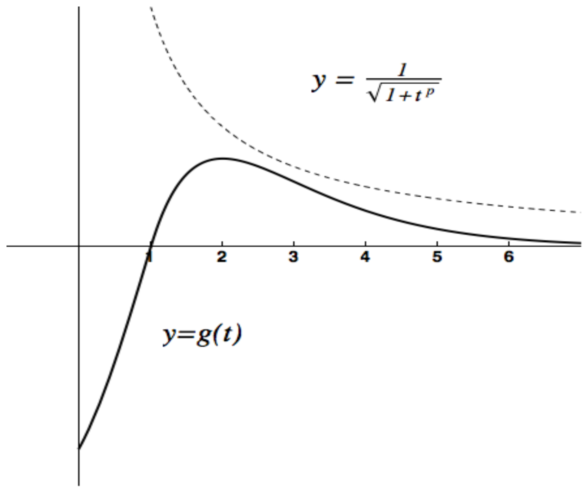
f. [2 points] The function $F(x) = \int_1^{x^2} \sin(t^2) dt$ is an even function.

True False

Solution: $F(-x) = \int_1^{(-x)^2} \sin(t^2) dt = \int_1^{x^2} \sin(t^2) dt = F(x)$.

2. [9 points] Consider the function $g(t)$, whose graph is shown below, which satisfies

$$0 < g(t) \le \frac{1}{\sqrt{1+t^p}} \qquad \text{for } t > 1.$$



a. [4 points] Find a formula for the antiderivative $f(t)$ of $g(t)$ satisfying $f(1) = 2$.

Solution: $f(t) = 2 + \int_1^t g(x)\,dx$

b. [2 points] For which values of $t \ge 0$ is $f(t)$ increasing?

Solution: $1 \le t \le \infty$

c. [2 points] For which values of $t \ge 0$ is $f(t)$ concave up?

Solution: $0 \le t \le 2$

d. [1 point] For which values of p is the limit $\lim_{t\to\infty} f(t)$ guaranteed to exist?

Solution: $p > 2$

3. [12 points] Let

$$I = \int_0^1 \left(1 + \frac{t^2}{2}\right)^{\frac{5}{2}} dt$$

a. [5 points] Approximate the value of I using Right(2) and Mid(2). Write each term in your sums.

Solution:

$$\begin{aligned} \text{Right}(2) &= \frac{1}{2}\left(\left(\frac{9}{8}\right)^{\frac{5}{2}} + \left(\frac{3}{2}\right)^{\frac{5}{2}}\right) \\ &\approx \frac{1}{2}(1.342 + 2.755) \approx 2.04904 \\ \text{Mid}(2) &= \frac{1}{2}\left(\left(\frac{33}{32}\right)^{\frac{5}{2}} + \left(\frac{41}{32}\right)^{\frac{5}{2}}\right) \\ &\approx \frac{1}{2}(1.08 + 1.858) \approx 1.46907 \end{aligned}$$

b. [2 points] Are your estimates of the value of I obtained using Right(2) and Mid(2) guaranteed to be overestimates, underestimates or neither?

Solution: Right = Overestimate (increasing)
Mid = Underestimate (concave up)

c. [3 points] Find the first three nonzero terms of the Taylor series for $g(t) = \left(1 + \frac{t^2}{2}\right)^{\frac{5}{2}}$ about $t = 0$.

Solution: Using the binomial series:

$$\begin{aligned} (1+x)^{\frac{5}{2}} &= 1 + \frac{5}{2}x + \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\frac{x^2}{2!} + \cdots \\ &= 1 + \frac{5}{2}x + \frac{15}{8}x^2 + \cdots \\ \left(1 + \frac{t^2}{2}\right)^{\frac{5}{2}} &= 1 + \frac{5}{2}\left(\frac{t^2}{2}\right) + \frac{15}{8}\left(\frac{t^2}{2}\right)^2 + \cdots \\ &= 1 + \frac{5}{4}t^2 + \frac{15}{32}t^4 + \cdots \end{aligned}$$

d. [2 points] Use your answer from part (c) to estimate I.

Solution:

$$I \approx \int_0^1 1 + \frac{5}{4}t^2 + \frac{15}{32}t^4 dt = t + \frac{5}{12}t^3 + \frac{3}{32}t^5\Big|_{t=0}^{1} = 1 + \frac{5}{12} + \frac{3}{32} = \frac{145}{96} \approx 1.510417$$

4. [11 points] A tank initially has 27 m^3 of water. At $t = 0$ (t in minutes), a pump takes water out of the tank. Let $V(t)$ be the volume of water (in m^3) in the tank t minutes after the pump was activated. Suppose the function $V(t)$ satisfies the differential equation

$$\frac{dV}{dt} = kV^{\frac{1}{3}}$$

where k is a constant.

a. [2 points] Is k positive or negative? What are the units of k?

Solution: $k < 0$, units are $\frac{\mathrm{m}^2}{\mathrm{min}}$

b. [7 points] Find a formula for $V(t)$. Your formula must contain only the constant k and the variable t.

Solution:

$$\frac{dV}{dt} = kV^{\frac{1}{3}}$$
$$\int V^{-\frac{1}{3}} dV = \int k dt$$
$$\frac{3}{2}V^{\frac{2}{3}} = kt + C$$
$$V = \left(\frac{2}{3}kt + C\right)^{\frac{3}{2}}$$
$$27 = V(0) = C^{\frac{3}{2}} \Rightarrow C = 27^{\frac{2}{3}} = 9$$
$$V(t) = \left(\frac{2}{3}kt + 9\right)^{\frac{3}{2}}$$

c. [2 points] How long does it take for the tank to empty? Your answer may contain the constant k.

Solution:

$$0 = V(t) = \left(\frac{2}{3}kt + 9\right)^{\frac{3}{2}}$$
$$0 = \frac{2}{3}kt + 9$$
$$t = -\frac{27}{2k}$$

5. [9 points] Consider the following power series:

$$\sum_{n=1}^{\infty} \frac{1}{n5^n}(x-4)^{n+1}$$

a. [4 points] Find the radius of convergence of the power series. Show all your work.

Solution:

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)5^{n+1}}}{\frac{1}{n5^n}} \frac{|x-4|^{n+2}}{|x-4|^{n+1}} = \frac{1}{5}|x-4| \lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{5}|x-4|.$$

$$\frac{1}{5}|x-4| < 1 \Leftrightarrow |x-4| < 5,$$

so $R = 5$.

b. [5 points] For which values of x does the series converge absolutely? For which values of x does it converge conditionally?

Solution: Converges absolutely inside radius: $(-1, 9)$.
Left endpoint: $x = -1$,

$$\sum_{n=1}^{\infty} \frac{1}{n5^n}(-5)^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

converges conditionally (alternating harmonic series).
Right endpoint: $x = 9$,

$$\sum_{n=1}^{\infty} \frac{1}{n5^n}5^{n+1} = 5\sum_{n=1}^{\infty} \frac{1}{n},$$

diverges. So, converges conditionally for $x = -1$, absolutely for $-1 < x < 9$.

6. [12 points] In the following problems, support all you answers by stating the test(s) or facts you used to prove convergence or divergence. Show all your work.

a. [4 points] $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^3}$ Circle your answer: Converges Diverges

Solution: Compare to: $\displaystyle\sum_{n=1}^{n} \frac{1}{n^{5/2}}$, converges by p-test, $p = \frac{5}{2} > 1$.

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{1+n^3}}{\frac{1}{n^{5/2}}} = \lim_{n\to\infty} \frac{n^3}{1+n^3} = 1 > 0,$$

so series converges by LCT.

b. [4 points] $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\cos^2(n)}$ Circle your answer: Converges Diverges

Solution: $\displaystyle\lim_{n\to\infty} \frac{1}{2+\cos^2(n)} \neq 0$ (does not exist, in fact), so series diverges.

c. [4 points] For which values of a does the series

$$\sum_{n=1}^{\infty} \frac{a^n}{3^n} = \frac{a}{3} + \frac{a^2}{9} + \frac{a^3}{27} + \cdots$$

converge? For the values of a where the series converges, find the sum of the series.

Solution: Geometric series, converges when $|r| = \left|\frac{a}{3}\right| < 1$, so converges on the interval $-3 < a < 3$. Converges to

$$\frac{a}{3}\left(\frac{1}{1-\frac{a}{3}}\right) = \frac{a}{3}\left(\frac{3}{3-a}\right) = \frac{a}{3-a}.$$

7. [14 points] You want to open a savings account to deposit 1000 dollars. Three banks offer the following options:

a. [3 points] Bank A offers its clients a savings account that earns 1.5% per year compounded annually. Define the sequence A_n to be the amount of money in the savings account n years after you deposit your 1000 dollars. Find a formula for A_n.

Solution: $A_n = 1000(1.015)^n$

b. [7 points] Bank B offers its clients a savings account that earns 2% per year compounded annually. At the end of each year, after the bank deposits the interest you earned, it withdraws a 1 dollar service fee from the account. Define the sequence B_n to be the amount of money, right after the service fee deduction, in the savings account n years after you deposit your 1000 dollars. Find B_1, B_2, B_3 and a **closed form** formula for B_n.

Solution:

$$
\begin{aligned}
B_1 &= 1000(1.02) - 1 = 1019. \\
B_2 &= (1000(1.02) - 1)(1.02) - 1 = 1000(1.02)^2 - (1 + 1.02) = 1038.38. \\
B_3 &= \big(1000(1.02)^2 - (1 + 1.02)\big)(1.02) - 1 = 1000(1.02)^3 - (1 + 1.02 + 1.02^2) \\
&= 1058.15. \\
&\vdots \\
B_n &= 1000(1.02)^n - (1 + 1.02 + 1.02^2 + \cdots + 1.02^{n-1}) = 1000(1.02)^n - \frac{1 - 1.02^n}{1 - 1.02}
\end{aligned}
$$

c. [4 points] Bank C offers its clients a savings account that earns interest continuously at a rate of 1.5% of the current balance per year. At the same time, the bank withdraws a service fee from the account at a rate of 1 dollar per year continuously. Let $M(t)$ be the amount of money in the savings account t years after you deposit your 1000 dollars. Write the differential equation satisfied by $M(t)$. Include initial conditions.

Solution: $\dfrac{dM}{dt} = 0.015M - 1, \qquad M(0) = 1000.$

8. [10 points] A solid S is obtained by rotating the region bounded by the curve $y = 1 - x^2$, the line $x = 0$, and the line $y = 0$ around the y-axis. The density of the solid is given by $\delta(y) = 1 + y$.

a. [5 points] Write a definite integral that gives the mass of the solid S.

Solution:

$$\int_0^1 \pi(1 - y)(1 + y)dy$$

b. [5 points] Find formulas for $\bar{x}$ and $\bar{y}$, the x and y coordinates of the center of mass of the solid S. The formulas may be written in terms of definite integrals, which you do not need to evalute.

Solution: $\bar{x} = 0$ by symmetry.

$$\bar{y} = \frac{\int_0^1 \pi y(1 - y)(1 + y)dy}{\int_0^1 \pi(1 - y)(1 + y)dy}$$

9. [11 points] An object is dropped from a height of 100 meters. If air resistance is considered, the height of the object $y(t)$ (in meters) above the ground t seconds after it was dropped is given by

$$y(t) = 100 - \frac{g}{k}t + \frac{g}{k^2}\left(1 - e^{-kt}\right).$$

where $k > 0$ is a constant representing the intensity of air resistance and $g = 9.8$ m/s^2 is the acceleration due to gravity.

a. [3 points] Show that $y(t)$ satisfies $y'' + ky' + g = 0$.

Solution:

$$\begin{aligned} y' &= -\frac{g}{k} + \frac{g}{k^2}ke^{-kt} = -\frac{g}{k}\left(1 - e^{-kt}\right) \\ y'' &= -\frac{g}{k}(ke^{-kt}) = -ge^{-kt} \\ y'' + ky' + g &= -ge^{-kt} - g(1 - e^{-kt}) + g = 0. \end{aligned}$$

b. [6 points] Use the first four nonzero terms of the Taylor series of the function $f(t) = e^{-kt}$ about $t = 0$ to find an approximation for $y(t)$.

Solution:

$$\begin{aligned} e^{-kt} &= 1 - kt + \frac{k^2}{2}t^2 - \frac{k^3}{6}t^3 + \cdots \\ y(t) &= 100 - \frac{g}{k}t + \frac{g}{k^2}\left(1 - 1 + kt - \frac{k^2}{2}t^2 + \frac{k^3}{6}t^3 - \cdots\right) \\ &= 100 - \frac{g}{2}t^2 + \frac{gk}{6}t^3 - \cdots \end{aligned}$$

c. [2 points] Using your answer from part (b), evaluate $\lim_{k\to 0} y(t)$.

Solution:

$$\lim_{k\to 0} y(t) = \lim_{k\to 0} 100 - \frac{g}{2}t^2 + \frac{gk}{6}t^3 - \cdots = 100 - \frac{g}{2}t^2.$$