Math 116 — First Midterm

October 9, 2013

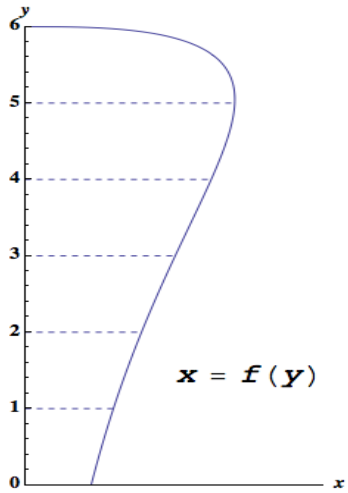
Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

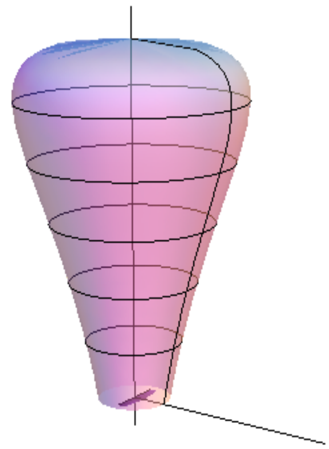
1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. All integrals must be computed by hand unless otherwise specified.
9. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	7	
2	12	
3	12	
4	17	
5	14	
6	11	
7	6	
8	10	
9	11	
Total	100	

1. [7 points] A lightbulb is obtained by revolving the curve $x = f(y)$ around the y-axis:



(a) Graph of $x = f(y)$



(b) 3D view of the bulb

The following table gives values of $x = f(y)$:

y	0	1	2	3	4	5	6
$x = f(y)$	0.8	1.1	1.5	1.9	2.4	2.8	0

a. [4 points] Write an integral involving $f(y)$ that computes the volume of the lightbulb.

Solution:

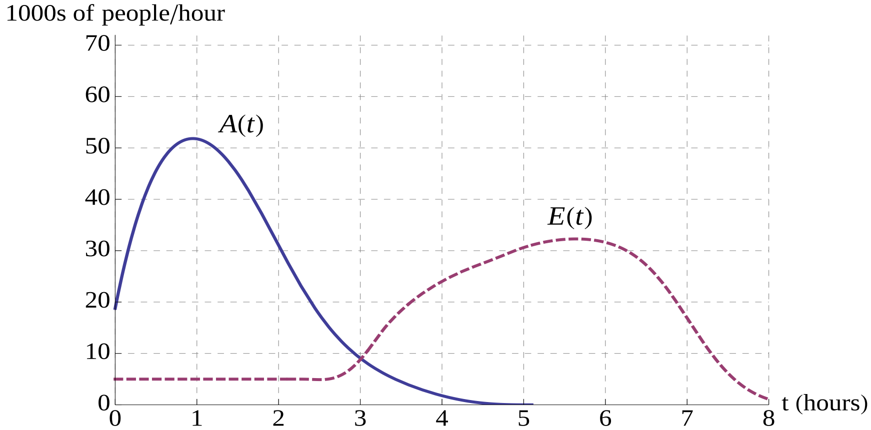
$$\int_0^6 \pi f(y)^2 dy$$

b. [3 points] Estimate the volume using the midpoint rule. Use the largest number of subintervals possible, given the information in the table above. Write out each of the terms in the sum.

Solution:

$$\text{MID}(3) = 2 \cdot \pi \cdot 1.1^2 + 2 \cdot \pi \cdot 1.9^2 + 2 \cdot \pi \cdot 2.8^2.$$

2. [12 points] The following is a graph showing the rates at which visitors arrive at and exit from the Big House during a football game:



The solid line $A(t)$ shows the arrival rate and the dashed line $E(t)$ shows the exit rate, in thousands of people per hour. Time t is measured in hours after 10 a.m. At 10 am, there are already 10,500 people inside the stadium.

a. [5 points] Let $P(t)$, the number of people, in thousands, in the stadium at time t. Give an expression for $P(t)$.

Solution:

$$P(t) = 10.5 + \int_0^t A(x) - E(x)dx.$$

b. [3 points] Give a practical interpretation of the statement $\int_1^4 A(t)dt = 67$. Include units.

Solution: 67,000 people arrived at the Big House between 11 a.m. and 2 p.m.

c. [2 points] At what time(s) of the day are there the most people in the stadium?

Solution: $t = 3$ (or 1 p.m).

d. [2 points] To comply with safety regulations, the game organizers must determine an upper bound (overestimate) of the number of people in the stadium at the beginning of the game. Circle the estimation rule(s) they could use. The game begins at noon.

left right trap **MID**

3. [12 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

a. [2 points] Let $F(x)$ be an antiderivative of a function $f(x)$. Then $F(2x)$ is an antiderivative of the function $f(2x)$.

True **[False]**

Solution: Let $f(x) = 3x^2$, then an antiderivative is $F(x) = x^3$ since $F'(x) = 3x^2 = f(x)$, but $F(2x) = (2x)^3 = 8x^3$ is not an antiderivative of $f(2x) = 3(2x)^2 = 12x^2$ since $\frac{d}{dx}(F(2x)) = \frac{d}{dx}(8x^3) = 24x^2 \neq f(2x) = 12x^2$.

b. [2 points] If $f(x)$ is a linear function on $[0, 1]$, then the midpoint rule computes the exact value of $\int_0^1 f(x)dx$.

[True] False

Solution:

c. [2 points] If $f(x)$ is a negative function that satisfies $f'(x) > 0$ for $0 \le x \le 1$. Then the right hand sums always yield an underestimate for the value of $\int_0^1 (f(x))^2 dx$.

[True] False

Solution: Let $g(x) = f(x)^2$, then $g'(x) = 2f(x)f'(x) < 0$ on $[0, 1]$. Since $g(x)$ is decreasing, then the right hand sum yields an underestimate for $\int_0^1 g(x)dx = \int_0^1 (f(x))^2 dx$

d. [2 points] If a and b are positive constants, then $\int e^{ax^2+b} dx = \frac{1}{2ax} e^{ax^2+b} + C$.

True **[False]**

Solution: Since $\frac{d}{dx}\left(\frac{1}{2ax} e^{ax^2+b}\right) \neq e^{ax^2+b}$, then the formula above is not true.

e. [2 points] The average value of $f(x)g(x)$ on an interval $[a, b]$ is the average value of $f(x)$ on $[a, b]$ times the average value of $g(x)$ on $[a, b]$.

True **[False]**

Solution: Let $f(x) = x$, $g(x) = 1-x$ and $[a, b] = [0, 1]$, then $\int_0^1 f(x)dx = \int_0^1 g(x)dx = \frac{1}{2}$, but $\int_0^1 f(x)g(x)dx = \int_0^1 x(1-x)dx = \frac{1}{6} \neq \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$.

f. [2 points] If $k > 0$ is a constant, the arclength of the function $y = kf(x)$ on an interval $[a, b]$ is k times the arclength of $y = f(x)$ on $[a, b]$.

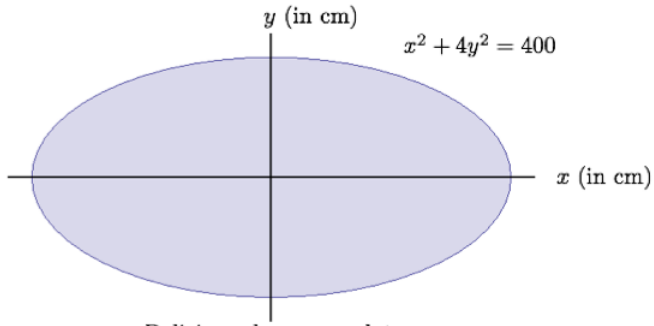
True **[False]**

Solution: Let $f(x) = 1$, $[a, b] = [0, 1]$ and $k = 2$, then the arclength of $f(x)$ on $[0, 1]$ is 1. The arclength of $y = 2f(x)$ on $[0, 1]$ is 1, not $2(1) = 2$.

4. [17 points]

a. [8 points] The delicious chicken shawarma platter is served on an elliptical plate, described by the equation $x^2 + 4y^2 = 400$. The mass density of the platter, including the food, is a function of y, given by $\delta(y) = 10 + 0.5y$ grams per cm^2.

In this problem, you do not need to evaluate any integrals.



Delicious shawarma plate

i) (4 points) Find an expression containing a definite integral that computes the mass of the chicken shawarma platter (including the food).

Solution:

$$m = \int_{-10}^{10} 2\sqrt{400 - 4y^2} \cdot (10 + 0.5y)dy.$$

ii) (4 points) Find expressions for the coordinates $\bar{x}$, $\bar{y}$ of the center of mass of the platter. If your expression does not involve an integral, include a justification.

Solution: $\bar{x} = 0$, since both the shape and density function are symmetric about the y-axis.

$$\bar{y} = \frac{\int_{-10}^{10} y \cdot 2\sqrt{400 - 4y^2} \cdot (10 + 0.5y)dy}{\int_{-10}^{10} 2\sqrt{400 - 4y^2} \cdot (10 + 0.5y)dy}.$$

b. [9 points] The mouthwatering kafta kabob platter is served on a circular plate, with radius 20 cm. Including the food, the overall mass density of the platter is given by $\delta(r) = \dfrac{50}{2+r^2}$ grams per cm^2, where r is the distance from the center of the plate (in cm).

i) (4 points) Write a definite integral that computes the mass of the kafta kabob platter (including food). You do not need to evaluate the integral.

Solution:

$$m = \int_0^{20} 2\pi r \cdot \frac{50}{2+r^2}\, dr.$$

ii) (3 points) Write an estimate for your expression in part i) of the mass of the platter using LEFT(3). Show all the terms in the sum. You do not need to evaluate the sum.

Solution:

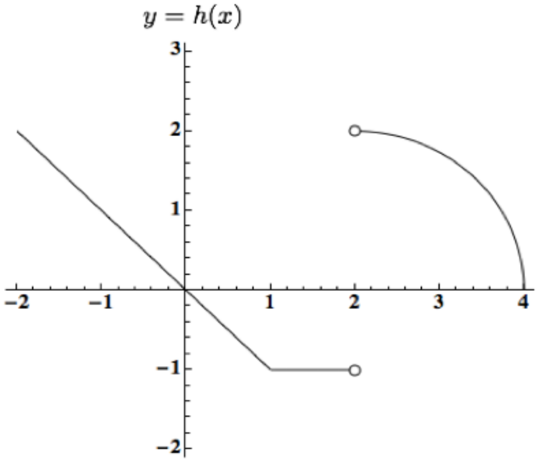
$$\begin{aligned} \text{LEFT}(3) &= \frac{20}{3}\left(2\pi \cdot 0 \cdot \frac{50}{2+0^2} + 2\pi \cdot \frac{20}{3} \cdot \frac{50}{2+(\frac{20}{3})^2} + 2\pi \cdot \frac{40}{3} \cdot \frac{50}{2+(\frac{40}{3})^2} \right) \\ &= \frac{20}{3}\left(0 + 45.09 + 23.30 \right). \\ &= \quad 0 + 300.6 + 155.33. \end{aligned}$$

iii) (2 points) Where is the center of mass of this platter? Justify.

Solution: At the center of the plate, since both the shape and density function are symmetric about the origin.

5. [14 points]

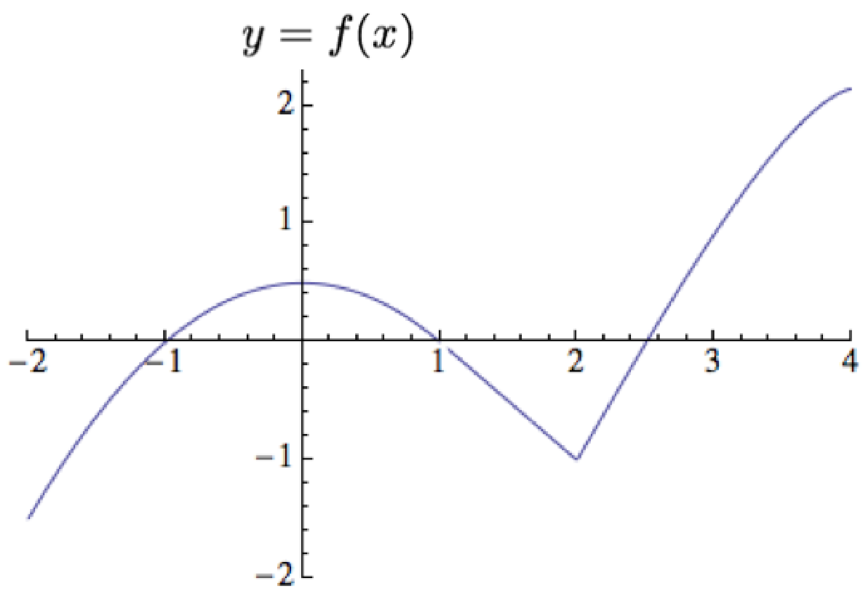
a. [5 points] The graph of the function $h(x)$, shown below, consists of line segments and a quarter of a circle.



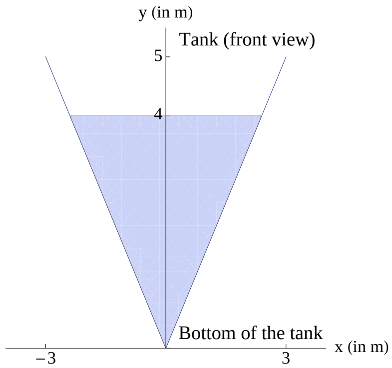
Let $f(x)$ be an antiderivative of $h(x)$ with $f(1) = 0$. Assume f is continuous. Fill in the table of values of $f(x)$, provided below, at the specified points.

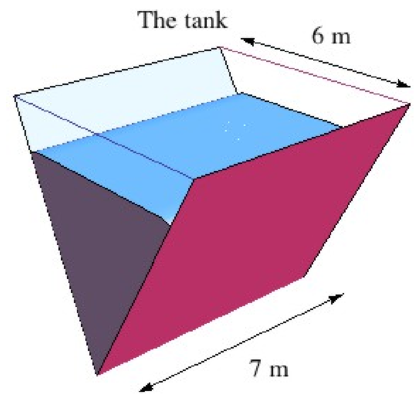
x	-2	-1	0	1	2	4
$f(x)$	-1.5	0	0.5	0	-1	$\pi - 1$

b. [9 points] Sketch the function $f(x)$. Make sure your graph indicates clearly where $f(x)$ is increasing, decreasing, concave up, and concave down, and appropriately reflects any critical points.



6. [11 points] The Math Department has recently acquired a triangular storage tank 6 m wide, 5 m tall and 7 m long, which it will use to store coffee for its graduate students. The tank currently contains a special coffee blend, with a mass density 1033 kg per m^3, up to a depth of 4 m.





a. [8 points] Write an expression that approximates the work done in lifting a horizontal slice of the liquid in the tank that is y meters above the bottom of the tank, with thickness Δy, to the top of the tank. Use $g = 9.8$ m per s^2 for the acceleration due to gravity.

Solution:

$$\Delta W = W_{slice} = \tfrac{6}{5}y \cdot 7 \cdot 9.8 \cdot 1033 \cdot (5 - y)\Delta y \ \text{ Joules.}$$

b. [3 points] While grading this exam, the grad students will need coffee. Find a definite integral that computes the work required to pump all the coffee to the top of the tank. Give the units of this integral. You do not need to evaluate it.

Solution:

$$W = \int_0^4 \tfrac{6}{5}y \cdot 7 \cdot 9.8 \cdot 1033 \cdot (5 - y)dy \ \text{ Joules.}$$

7. [6 points] Let $g(t)$ be the concentration of caffeine (in milligrams per liter) in the bloodstream of a Math 116 GSI, t hours after calculus exam grading begins. Define

$$G(t) = \int_0^t g(x)dx,$$

and let $A(t) = \frac{1}{t}\, G(t)$.

a. [3 points] What is the practical interpretation of the statement $A(4) = 70$? Include units.

Solution: The average concentration of caffeine in the bloodstream of a Math 116 GSI, during the first 4 hours of calculus exam grading, is 70 milligrams per liter.

b. [3 points] Find $A'(t)$.

Solution: By the product rule and the fundamental theorem of calculus,

$$A'(t) = \frac{d}{dt}\left(\frac{1}{t}G(t)\right) = -\frac{1}{t^2}\int_0^t g(x)dx + \frac{1}{t}g(t).$$

Some alternate (equivalent) answers:

$$A'(t) = -\frac{1}{t^2}G(t) + \frac{1}{t}g(t), \qquad A'(t) = -\frac{1}{t}A(t) + \frac{1}{t}g(t).$$

8. [10 points] Let $g(x)$ be an odd continuous function and $h(x)$ an antiderivative of $g(x)$. Find the values of the following expressions given the values of the functions below.

x	0	1	2	3	4
$g(x)$	0	5	1	-2	-1
$h(x)$	2	6	10	8	5

Show all your work to receive full credit.

a. [3 points] $\displaystyle\int_{-1}^{2}(3g(t)-5)dt$

Solution:

$$\int_{-1}^{2}(3g(t)-5)dt = 3\int_{-1}^{2}g(t)dt - 5\int_{-1}^{2}dt$$

since $g(t)$ is odd $\displaystyle\int_{-1}^{1}g(t)dt = 0.$

$$\begin{aligned} 3\int_{-1}^{2}g(t)dt - 5\int_{-1}^{2}dt &= 3\int_{1}^{2}g(t)dt - 5(3) \\ &= 3(h(2)-h(1)) - 15 \\ &= 3(10-6) - 15 = 12 - 15 = -3. \end{aligned}$$

b. [4 points] $\displaystyle\int_{1}^{3}tg'(t)dt$

Solution: Using integration by parts with $u = t$, $v' = g'(t)$, then $u' = 1$ and$v = g(t)$,

$$\begin{aligned} \int_{1}^{3}tg'(t)dt &= tg(t)\,\Big|_1^3 - \int_{1}^{3}g(t)dt \\ &= 3g(3) - g(1) - (h(3)-h(1)) \\ &= 3(-2) - 5 - (8-6) = -6 - 5 - 2 = -13. \end{aligned}$$

c. [3 points] Let $F(x) = \displaystyle\int_{3}^{4x}h(t)dt$. Find $F'(1)$.

Solution: Using the second Fundamental Theorem of Calculus,

$$F'(x) = 4h(4x) \qquad \text{then} \qquad F'(1) = 4h(4) = 20.$$

9. [11 points] Consider the region R bounded by $y = 2(1 - e^{-x})$, $y = 1$ and $x = 3$. In this problem, you do not need to evaluate the integrals.

a. [4 points] Find a definite integral that computes the volume of the solid obtained by rotating the region R about the y-axis.

Solution:

•Washer method:

$$V = \int_1^{2(1-e^{-3})} \pi \left(3^2 - \left(-\ln\left(1 - \frac{y}{2}\right)\right)^2\right) dy$$

$$= \int_1^{2(1-e^{-3})} \pi \left(9 - \ln^2\left(1 - \frac{y}{2}\right)\right) dy$$

•Shell method:

$$V = \int_{\ln 2}^{3} 2\pi x \left(2(1 - e^{-x}) - 1\right) dx$$

b. [7 points] Find a definite integral that computes the volume of the solid with base given by the region R, and whose cross sections perpendicular to the x-axis are semicircles.

Solution:

$$V = \int_{\ln 2}^{3} \frac{1}{2}\left(\pi\left(\frac{1}{2}\left[2(1 - e^{-x}) - 1\right]\right)^2\right) dx$$

$$= \int_{\ln 2}^{3} \frac{\pi}{8}\left[2(1 - e^{-x}) - 1\right]^2 dx$$