Math 116 — Second Midterm

November 13, 2013

Name: EXAM SOLUTIONS

Instructor: ____________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. All integrals must be computed by hand unless otherwise specified.
9. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	10	
2	12	
3	14	
4	12	
5	10	
6	12	
7	8	
8	15	
9	7	
Total	100	

1. [10 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

a. [2 points] If $r = f(\theta)$ is a polar curve and is concave down, then $f''(\theta) < 0$.

True $\boxed{\text{False}}$

Solution: The function $f(\theta) = 1$ is concave down for $0 < \theta < \pi$, but $f''(\theta) = 0$.

b. [2 points] Let $y = f(x)$ be a solution of the differential equation $y' = g(x)$ where $g(x)$ is an increasing function. Then the graph of $f(x)$ is concave up.

$\boxed{\text{True}}$ False

Solution: Since $y'' = g'(x) > 0$, then $f(x)$ is concave up.

c. [2 points] The function $x(t) = e^{-3t} + 2t^2 + \frac{4}{9}$ is a solution to $x'' = 9x - 18t^2$.

$\boxed{\text{True}}$ False

Solution: $x'' = 9e^{-3t} + 4$ and $9x - 18t^2 = 9(e^{-3t} + 2t^2 + \frac{4}{9}) - 18t^2 = 9e^{-3t} + 4$. Hence $x'' = 9x - 18t^2$.

d. [2 points] If $\int_0^\infty f(x)\,dx$ and $\int_0^\infty g(x)\,dx$ both diverge, then $\int_0^\infty f(x)g(x)dx$ diverges.

True $\boxed{\text{False}}$

Solution: If $f(x) = g(x) = \frac{1}{x+1}$, then
$\int_0^\infty f(x)\,dx = \int_0^\infty g(x)\,dx = \int_0^\infty \frac{1}{x+1}\,dx = \lim_{b\to\infty} \ln|x+1||_0^b = \infty$ (diverges), but
$\int_0^\infty f(x)g(x)dx = \int_0^\infty \frac{1}{(1+x)^2}dx = \lim_{b\to\infty} -\frac{1}{1+x}|_0^b = 1$ (converges).

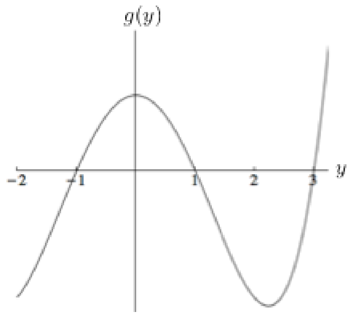
e. [2 points] If $k > 0$ is a constant, then on the interval $a \le t \le b$, the arclength of the parametric curve $x = kf(t)$, $y = kg(t)$ is k times the arclength of $x = f(t), y = g(t)$.

$\boxed{\text{True}}$ False

Solution: $\int_a^b \sqrt{\left(\frac{d(kf(t))}{dt}\right)^2 + \left(\frac{d(kg(t))}{dt}\right)^2}dt = k\int_a^b \sqrt{\left(\left(\frac{d(f(t))}{dt}\right)^2 + \left(\frac{d(g(t))}{dt}\right)^2\right)}dt$

2. [12 points]

a. [10 points] Suppose the function $y(t)$ satisfies the differential equation $\frac{dy}{dt} = g(y)$, where the graph of $g(y)$ is shown below:



1.(4 pts) Use inequalities to describe the regions in the y-t plane where the solution curves of the differential equation are strictly increasing.

Solution: $y(t)$ is increasing on $-1 < y < 1$ and $3 < y$ since $\frac{dy}{dt} = g(y) > 0$ in these intervals.

2.(6 pts) Find all equilibrium solutions (if any) to the differential equation for $y(t)$. Classify each one as stable or unstable. If the equation does not have equilibrium solutions, write none.

Solution: $y = -1$ and $y = 3$ are unstable, $y = 1$ is stable.

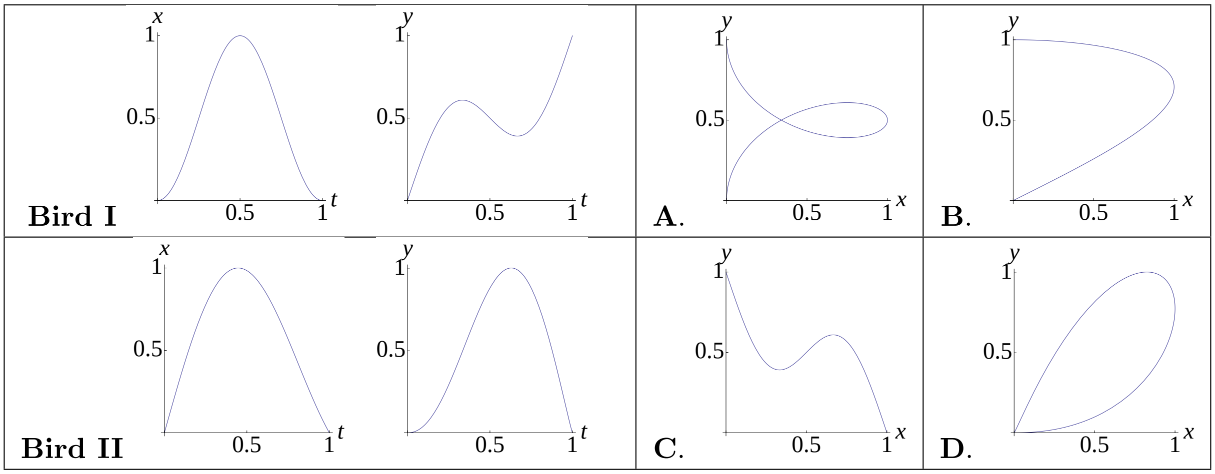
b. [2 points] Consider the differential equation

$$\frac{dy}{dt} = (2y + 5t)t.$$

Find all equilibrium solutions (if any) to the differential equation for y. If the equation does not have equilibrium solutions, write none.

Solution: None

3. [14 points] The x and y positions of two birds in flight, Bird I and Bird II, are graphed below as functions of time t (see figures labeled Bird I and Bird II on the left). To the right, there are four parametric curves, A,B,C,D, showing flight paths of several birds in the x-y plane.



a. [2 points] Is the horizontal velocity of bird I zero at any time $0 < t < 1$? If so, give an approximate t value.

Solution: $t = 0.5$

b. [2 points] Based on the plots shown for bird II, consider a parametric curve for the flight path for bird II in the x-y plane. Would the slope of the tangent line to the flight path curve at time $t = 0.9$ be positive, negative, or zero? Justify.

Solution: Positive slope since $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$ and $x'(0.9)$ and $y'(0.9)$ both are negative.

c. [4 points] One of the parametric curves A,B,C,D corresponds to bird I and another corresponds to bird II. Indicate which ones by circling the correct answers:

Solution:

Bird I corresponds to:	**A**	b	c	d
Bird II corresponds to:	a	b	c	**D**

d. [6 points] A third bird flies according to the following parametric equations

$$x(t) = 1 - t^3 \qquad y(t) = t^2 - t.$$

1. Find the time(s) at which the bird is flying straight horizontally right or left. Show all your work.

Solution: The bird is flying straight horizontally right or left if $y'(t) = 2t - 1 = 0$. Hence $t = \frac{1}{2}$.

2. Find the speed of the bird at $t = 1$. Show all your work.

Solution: Speed$= \sqrt{\left(\frac{d(1-t^3)}{dt}\right)^2 + \left(\frac{d(t^2-t)}{dt}\right)^2} = \sqrt{(-3t^2)^2 + (2t-1)^2}$.
At $t = 1$, speed$= \sqrt{9+1} = \sqrt{10}$.

4. [12 points] A volcano has erupted, covering its surroundings with ash. The area covered with ash is bounded by the polar curve $r = \dfrac{2}{2+\sin(\theta+\frac{\pi}{4})}$, where the volcano is located at the origin and r is measured in km (see the shaded region in Figure 1).

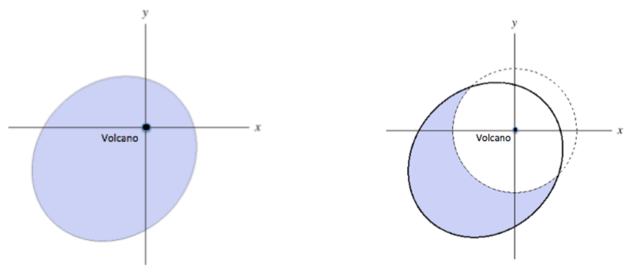


Figure 1. Affected area. Figure 2. Area to be cleaned.

a. [3 points] Local authorities want to estimate the area of the region covered with ash. Write an integral that computes the total area covered with ash. You do not need to compute the value of the integral.

Solution:

$$A = \int_0^{2\pi} \frac{1}{2}\left(\frac{2}{2+\sin(\theta+\frac{\pi}{4})}\right)^2 d\theta.$$

b. [7 points] Days after the eruption, the government designs a plan to remove the ashes **outside** a circle of radius one km around the volcano (see the shaded region in Figure 2). Find an expression containing definite integrals that computes the cost of cleaning this region, assuming it costs 3 million dollars to clean the ashes in one km square of land. You do not need to compute the value of the integral.

Solution: The curves intersect when θ satisfies $1 = \dfrac{2}{2+\sin(\theta+\frac{\pi}{4})}$. This equation leads to $\sin(\theta+\frac{\pi}{4}) = 0$. Then $\theta = \frac{3}{4}\pi$ and $\theta = \frac{7}{4}\pi$.

$$\text{Cost} = \$3 \text{ million} \cdot \left(\int_{\frac{3}{4}\pi}^{\frac{7}{4}\pi} \frac{1}{2}\left(\frac{2}{2+\sin(\theta+\frac{\pi}{4})}\right)^2 d\theta - \int_{\frac{3}{4}\pi}^{\frac{7}{4}\pi} \frac{1}{2} d\theta \right).$$

or

$$\text{Cost} = \$3 \text{ million} \cdot \left(\int_{\frac{3}{4}\pi}^{\frac{7}{4}\pi} \frac{1}{2}\left(\frac{2}{2+\sin(\theta+\frac{\pi}{4})}\right)^2 d\theta - \frac{\pi}{2} \right).$$

c. [2 points] How far from the volcano is the farthest point covered with ash? Justify your answer.

Solution: Since $r = \dfrac{2}{2+\sin(\theta+\frac{\pi}{4})}$ and $-1 \le \sin(\theta+\frac{\pi}{4}) \le 1$, then $\dfrac{2}{3} = \dfrac{2}{2+1} \le \dfrac{2}{2+\sin(\theta+\frac{\pi}{4})} \le \dfrac{2}{2-1} = 2$. Hence the the farthest point covered with ash is 2 km away from the volcano.

5. [10 points] Consider a group of people who have received a new treatment for pneumonia. Let t be the number of days it takes for a person with pneumonia to fully recover. The probability density function giving the distribution of t is

$$f(t) = \frac{10}{(1+at)^2}, \qquad \text{for } t > 0,$$

for some positive constant a.

a. [2 points] Give a practical interpretation of the quantity $\int_3^{10} f(t)dt$. You do not need to compute the integral.

Solution: The fraction of the people who recovered in between three and ten days after the treatment.

b. [5 points] Find a formula for the cumulative distribution function $F(t)$ of $f(t)$ for $t > 0$. Show all your work. Your answer may include a, but it should not contain any integrals.

Solution: $F(t) = \int_0^t \frac{10}{(1+ax)^2}dx = -\frac{10}{a(1+ax)}\Big|_0^t = \frac{10}{a} - \frac{10}{a(1+at)}$

c. [3 points] Determine the value of a. Show all your work.

Solution: Since $f(t)$ is a probability density function, then $1 = \int_0^\infty \frac{10}{(1+ax)^2}dx$. Hence

$$\int_0^\infty \frac{10}{(1+ax)^2}dx = \lim_{b\to\infty} \int_0^b \frac{10}{(1+ax)^2}dx = \lim_{b\to\infty} \frac{10}{a} - \frac{10}{a(1+ab)} = \frac{10}{a}.$$

Hence $a = 10$.

6. [12 points] Determine the convergence or divergence of the following improper integrals. Justify your answers. Make sure to properly cite any results of convergence or divergence of integrals that you use. If you use the comparison test, be sure to show all your work. Circle your answer.

a. [4 points] $\displaystyle\int_3^\infty \frac{1}{\sqrt[3]{x}+e^{2x}}dx.$ **CONVERGES** DIVERGES

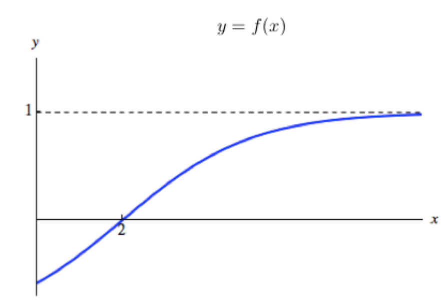
Solution: Since
$$\frac{1}{\sqrt[3]{x}+e^{2x}} \leq \frac{1}{e^{2x}} = e^{-2x}$$
and $\displaystyle\int_3^\infty e^{-2x}dx$ converges then $\displaystyle\int_3^\infty \frac{1}{\sqrt[3]{x}+e^{2x}}dx$ converges.

b. [4 points] $\displaystyle\int_2^\infty \frac{3+b\sin^2(x^4)}{x^5}dx$, where b is a positive constant.

CONVERGES DIVERGES

Solution: Since
$$\frac{3+b\sin^2(x^4)}{x^5} \leq (3+b)\left(\frac{1}{x^5}\right)$$
and $\displaystyle\int_2^\infty \frac{1}{x^5}dx$ converges ($p>1$) then $\displaystyle\int_2^\infty \frac{3+b\sin^2(x^4)}{x^5}dx$ converges.

c. [4 points] Let $f(x)$ be the differentiable function shown below. Note that $f(x)$ has a horizontal asymptote at $y=1$.



Does $\displaystyle\int_2^\infty \frac{f'(x)}{1+f(x)}dx$ converge or diverge? Circle your answer. If it converges, find its value.

CONVERGES DIVERGES

Solution:
$$\int_2^\infty \frac{f'(x)}{1+f(x)}dx = \lim_{b\to\infty}\int_2^b \frac{f'(x)}{1+f(x)}dx$$
$$= \lim_{b\to\infty} \ln|1+f(x)|\Big|_2^b = \lim_{b\to\infty} \ln|1+f(b)| - \ln|1+f(2)| = \ln 2.$$

7. [8 points] The population of termites $P(t)$ (in thousands) in a tree grows at a rate $f(P)$, in thousands of termites per day. A pesticide is applied to the tree to eliminate the termites. As a result, the population of termites $P(t)$ satisfies

$$\frac{dP}{dt} = f(P) - 3e^{-\frac{1}{3}t},$$

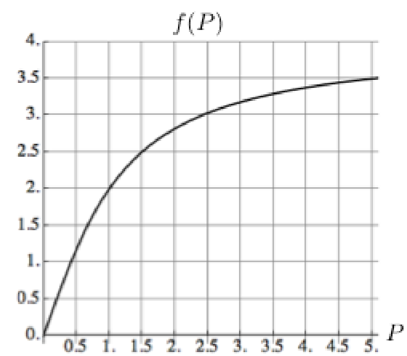
where t is measured in days since the pesticide is applied.

a. [4 points] Use Euler's method with steps of $\Delta t = 0.5$ to estimate the amount of termites in the tree one day after the pesticide is applied. It is estimated that there are 2500 termites in the tree at the time the pesticide is applied ($t = 0$). The graph of $f(P)$ is given below. Show all your computations.

Solution:

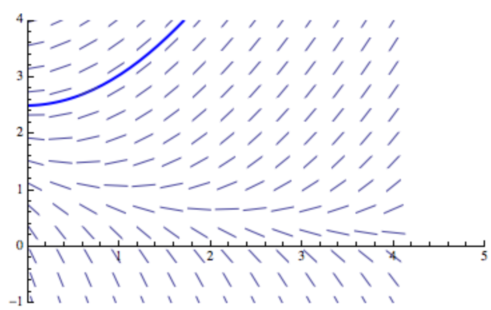
$$\begin{aligned} P(0) &= 2.5 \\ P(0.5) &\approx 2.5 + (0.5)(3 - 3e^{-\frac{1}{3}(0)}) = 2.5 \\ P(1) &\approx 2.5 + (0.5)(3 - 3e^{-\frac{1}{3}(0.5)}) = 2.73. \end{aligned}$$

t	0	0.5	1
$P(t)$	2.5	2.5	2.73



b. [2 points] The slope field of the differential equation satisfied by $P(t)$ is shown below. Sketch the graph of $P(t)$.

Solution:



c. [2 points] Is the estimate obtained in part (**a**) guaranteed to be an overestimate or an underestimate? Justify.

Solution: Underestimate because $P(t)$ is concave up.

8. [15 points] Two zombies are chasing Jake down the Diag. Let $J(t)$ be Jake's position, measured in meters along the Diag, as he runs from the zombies. In this problem the time t is measured in seconds.

a. [3 points] The velocity of the first zombie is proportional to the difference between its own position, $S(t)$, and Jake's position, with constant of proportionality k. Using this fact, write the differential equation satisfied by $S(t)$.

Solution:

$$\frac{dS}{dt} = k(S - J(t)).$$

b. [2 points] State whether your equation in part (a) is separable. Circle the correct answer.

Solution:

The equation is: separable **NOT SEPARABLE**

Note: $J(t)$ is not constant, since Jake is running.

c. [9 points] The position of the second zombie at time t is given by the function $Z(t)$ (in meters), and satisfies the differential equation

$$\frac{dZ}{dt} = \alpha \frac{J(t)}{Z},$$

where α is a positive constant. Assuming that $Z(0) = 5$ and that Jake's position is given by $J(t) = 2t + 10$, find a formula for $Z(t)$.

Solution: Separating gives:

$$\int Z dZ = \alpha \int 2t + 10 \; dt,$$

and so

$$\tfrac{1}{2}Z^2 = \alpha(t^2 + 10t) + C.$$

Plugging in $Z(0) = 5$, we see that $C = \frac{25}{2}$, so $Z(t)$ is given by:

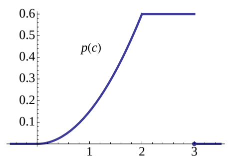
$$Z(t) = \sqrt{2\alpha t^2 + 20\alpha t + 25}.$$

d. [1 point] In the differential equation $\dfrac{dZ}{dt} = \alpha \dfrac{J(t)}{Z}$, what are the units of α?

Solution: The units are m/s.

9. [7 points]
Thanks to the Math Department's acquisition of a coffee tank in October, there are now 300 cups of coffee available to the graduate students each day.
The department wants to assess how much of the coffee is drunk and how much is wasted. Let c be the amount of coffee drunk in one day, measured in hundreds of cups of coffee. The probability density function for c is given by

$$p(c) = \begin{cases} \frac{3}{20}c^2 & \text{for } 0 \le c \le 2 \\ \frac{3}{5} & \text{for } 2 \le c \le 3 \\ 0 & \text{otherwise.} \end{cases}$$



a. [4 points] Find the mean of the amount of coffee drunk in one day. Include units. Show all your work.

Solution: The mean is
$$\int_{-\infty}^{\infty} c\ p(c)dc = \int_0^3 c\ p(c)dc.$$
Since $p(c)$ is defined piecewise, we break the integral into the two pieces:
$$\begin{aligned}\int_0^3 cp(c)dc &= \int_0^2 c \cdot \tfrac{3}{20}c^2 dc + \int_2^3 c \cdot \tfrac{3}{5} dc \\ &= \tfrac{3}{80}c^4\Big|_0^2 + \tfrac{3}{10}c^2\Big|_2^3 = \tfrac{3}{80}\cdot(16-0) + \tfrac{3}{10}(9-4) \\ &= 2.1 \text{ or } \tfrac{168}{80}.\end{aligned}$$
So, the mean amount of coffee drunk in one day is **210 cups of coffee**.

b. [3 points] Find the median of the amount of coffee drunk in one day. Include units. Show all your work.

Solution: We want to find M such that
$$\int_0^M p(c)dc = \int_M^3 p(c)dc = \frac{1}{2}.$$
From the graph, we see that the area between $c = 2$ and $c = 3$ is $\frac{3}{5}$ (since it is rectangular), which is already more than $\frac{1}{2}$. So, the median will be in the interval $2 \le c \le 3$, and we can use the second part of the piecewise formula:
$$\tfrac{1}{2} = \int_M^3 \tfrac{3}{5}dc = \tfrac{3}{5}(3 - M),$$
so $M = 3 - \frac{5}{6} \approx 2.17$. So, the median amount of coffee drunk in one day is **217 cups**.

(**Note**: the area in the first interval is $\displaystyle\int_0^2 \tfrac{3}{20}c^2dc = \tfrac{1}{20}\cdot 2^3 = \tfrac{2}{5}$, which is 0.1 less than $\frac{1}{2}$. So, we could instead solve $\displaystyle\int_2^M \tfrac{3}{5}dc = 0.1$.)