Math 116 — Final Exam

December 17, 2013

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 14 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. All integrals must be computed by hand unless otherwise specified.
9. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	8	
2	11	
3	8	
4	9	
5	12	
6	6	
7	8	
8	12	
9	13	
10	13	
Total	100	

You may find the following expressions useful.

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n \, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n \, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

1. [8 points] Consider the differential equation and initial condition

$$\frac{dy}{dt} = At - y, \qquad y(0) = 5,$$

where $A > 0$ is a constant. This differential equation is not separable, but it is still possible to solve it using the following steps.

a. [5 points] Let $w(t) = \frac{dy}{dt}$. If you differentiate both sides of the differential equation above with respect to t, you obtain that the function $w(t)$ satisfies

$$\frac{dw}{dt} = A - w.$$

Find a general formula for $w(t)$, showing all work. Your answer may include A.

Solution:

$$\begin{aligned} \frac{dw}{dt} &= A - w \\ \frac{dw}{A - w} &= dt \\ -\ln|A - w| &= t + C \\ w(t) &= A - Be^{-t} \end{aligned}$$

b. [1 point] Given that $\frac{dy}{dt} = At - y$ and $y(0) = 5$, what must be the value of $w(0)$? Your answer may include A.

Solution: $w(0) = \frac{dy}{dt}|_{t=0} = At - y(t)|_{t=0} = 0 - 5 = -5.$

c. [2 points] Use the definition of $w(t)$ to obtain a formula for $y(t)$. Your answer may include A.

Solution:

$$y(t) = \int A - Be^{-t} dx = At + Be^{-t} + C$$

or using the differential equation

$$y(t) = At - \frac{dy}{dt} = At - (A - Be^{-t}) = A(t-1) + Be^{-t}.$$

2. [11 points] Determine the convergence or divergence of the following series. In parts (a) and (b), support your answers by stating and properly justifying any test(s), facts or computations you use to prove convergence or divergence. Circle your final answer. Show all your work.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty} \frac{9n}{e^{-n}+n}$ CONVERGES **DIVERGES**

Solution:

$$\lim_{n\to\infty} \frac{9n}{e^{-n}+n} = \lim_{n\to\infty} \frac{9n}{n} = 9 \neq 0.$$

Since $\lim_{n\to\infty} a_n \neq 0$ then $\displaystyle\sum_{n=1}^{\infty} \frac{9n}{e^{-n}+n}$ diverges.

b. [4 points] $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n(\ln n)^2}$. **CONVERGES** DIVERGES

Solution: The function $f(n) = \frac{4}{n(\ln n)^2}$ is positive and decreasing for $n > 2$, then by Integral Test the convergence or divergence of $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n(\ln n)^2}$ can be determined with the convergence or divergence of $\displaystyle\int_2^{\infty} \frac{4}{x(\ln x)^2}dx$

$$\int \frac{4}{x(\ln x)^2}dx = \int \frac{4}{u^2}du \qquad \text{where } u = \ln x.$$

$$= -\frac{4}{u} + C = -\frac{4}{\ln x} + C$$

Hence

$$\int_2^{\infty} \frac{4}{x(\ln x)^2}dx = \lim_{b\to\infty} -\frac{4}{\ln x}\Big|_2^b = -\frac{4}{\ln 2} \qquad \text{converges.}$$

or

$$\int_2^{\infty} \frac{4}{x(\ln x)^2}dx = 4\int_{\ln 2}^{\infty} \frac{1}{u^2}du \qquad \text{converges by } p\text{-test with } p = 2 > 1.$$

c. [4 points] Let r be a **real** number. For which values of r is the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^r+4}$ absolutely convergent? Conditionally convergent? No justification is required.

Solution:

Absolutely convergent if : $r > 3$

Conditionally convergent if : $2 < r \leq 3$

Justification (not required):

- Absolute convergence:
  The series $\sum_{n=1}^{\infty}\left|(-1)^n \frac{n^2}{n^r+4}\right| = \sum_{n=1}^{\infty}\frac{n^2}{n^r+4}$ behaves like $\sum_{n=1}^{\infty}\frac{n^2}{n^r} = \sum_{n=1}^{\infty}\frac{1}{n^{r-2}}$. The last series is a p-series with $p = r-2$ which converges if $r-2 > 1$. Hence the series converges absolutely if $r > 3$.

- Conditionally convergence:
  The function $\frac{n^2}{n^r+4}$ is positive and decreasing (for large values of n) when $r > 2$. Hence by the Alternating series test $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^r+4}$ converges in this case.

3. [8 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

a. [2 points] The equation $y^3 - x^3 = xy$ in Cartesian coordinates can be written in polar coordinates as

$$r = \frac{\sin\theta\cos\theta}{\sin^3\theta - \cos^3\theta}.$$

[True] False

Solution: Let $x = r\cos\theta$ and $y = r\sin\theta$, then $y^3 - x^3 = (r\sin\theta)^3 - (r\cos\theta)^3 = r^3(\sin^3\theta - \cos^3\theta)$, $xy = r^2\sin\theta\cos\theta$, then $r = \dfrac{\sin\theta\cos\theta}{\sin^3\theta - \cos^3\theta}$.

b. [2 points] If $g(x) = \int_1^x f(t)dt$, then $g(4) - g(2) = \int_2^4 f(t)dt$.

[True] False

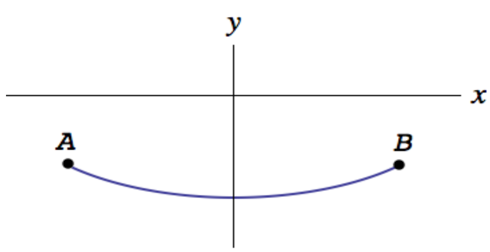
Solution: Since $g'(x) = f(x)$, then $g(x)$ is an antiderivative of $f(x)$. By the Fundamental Theorem of Calculus $\int_2^4 f(t)dt = g(4) - g(2)$.

c. [2 points] The function $h(x) = \int_0^{\sin x} e^{-t^2}dt$ has a local maximum at $x = \frac{\pi}{2}$.

[True] False

Solution: Since $h'(x) = e^{-(\sin x)^2}\cos x$ and $h'(\frac{\pi}{2}) = 0$ and $h'(x)$ changes signs from positive to negative at $x = \frac{\pi}{2}$. Hence $h(x)$ has a local maximum at $x = \frac{\pi}{2}$.

d. [2 points] The graph of the parametric equations $x = f(t)$ and $y = f'(t)$ for some function $f(t)$ is shown below:



As t increases, the curve is traced from A to B.

True [False]

Solution: Since the graph is below the y-axis, then $y = f'(t) < 0$. Hence $f(t)$ is decreasing, and since $x = f(t)$, then as t increases, the values of x decreases. Hence the curve is traced from B to A.

4. [9 points]
Determine if each of the following sequences is increasing, decreasing or neither, and whether it converges or diverges. If the sequence converges, identify the limit. Circle all your answers. No justification is required.

a. [3 points] $a_n = \displaystyle\int_1^{n^3} \frac{1}{(x^2+1)^{\frac{1}{5}}}\,dx.$

Converges to ________________ **DIVERGES**.

INCREASING Decreasing Neither.

Solution: (Not required)
Since $\displaystyle\int_1^{\infty} \frac{1}{(x^2+1)^{\frac{1}{5}}}\,dx \approx \int_1^{\infty} \frac{1}{x^{\frac{2}{5}}}\,dx$ which diverges by p-test (with $p = \frac{2}{5} \leq 1$). Hence, $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \int_1^{n^3} \frac{1}{(x^2+1)^{\frac{1}{5}}}\,dx = \int_1^{\infty} \frac{1}{(x^2+1)^{\frac{1}{5}}}\,dx = \infty$. The sequence is increasing since $\displaystyle\frac{1}{(x^2+1)^{\frac{1}{5}}} > 0$.

b. [3 points] $b_n = \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}.$

CONVERGES TO $\sin 1$ Diverges.

Increasing Decreasing **NEITHER**.

Solution: (Not required)
Since $\sin x = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, then $\displaystyle\lim_{n\to\infty} b_n = \sum_{k=0}^{\infty} \frac{(-1)^k (1)^{2k+1}}{(2k+1)!} = \sin 1$. Since the series is alternating, the sequence is neither increasing or decreasing.

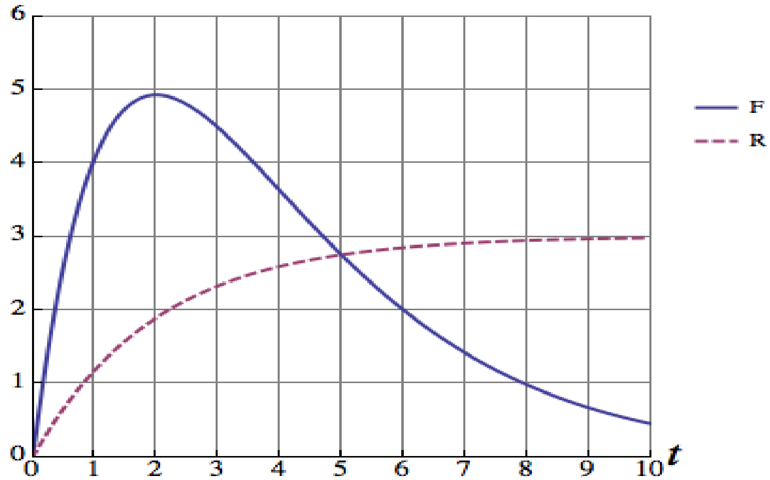
c. [3 points] $c_n = \cos(a^n)$, where $0 < a < 1$.

CONVERGES TO 1 Diverges.

INCREASING Decreasing Neither.

Solution: (Not required)
Since $\displaystyle\lim_{n\to\infty} a^n = 0$, then $\displaystyle\lim_{n\to\infty} \cos a^n = \cos 0 = 1$.

5. [12 points] The motor inside a submersible produces toxic fumes at a rate of $F(t)$ mg per minute, t minutes after the motor is activated. The submersible is equipped with an air filter system that removes the fumes at a rate $R(t)$ mg per minute. The total volume of air inside the submersible is 10 m^3, and remains constant. The graph of $F(t)$ (solid line) and $R(t)$ (dashed line) are shown below.



a. [4 points] Estimate the quantity of toxic fumes (in mg) produced by the motor during the first 4 minutes using the Midpoint rule with 2 subintervals. Write all the terms in your sum.

Solution: Toxic fumes $\approx 2(4 + 4.5) = 17$ mg

b. [2 points] Is your estimate guaranteed to be an overestimate, underestimate or is it not possible to tell? Justify.

Solution: It is an overestimate since the $F(t)$ is concave down in $[0, 4]$.

c. [2 points] At what time is the quantity of fumes in the submersible greatest?

Solution: $t = 5$ minutes.

d. [4 points] Let $C(t)$ be the *concentration* of the fumes (in mg per m^3) in the air inside the submersible, t minutes after the motor is activated. Find a formula for $C(t)$.

Solution:

$$C(t) = \frac{1}{10}\int_0^t F(x) - R(x)dx.$$

6. [6 points] Consider the following differential equation

$$\frac{dy}{dx} = (x - y)(y - 2)$$

a. [2 points] Find all the equilibrium solutions of the differential equation (if any). If the differential equation has no equilibrium solutions, write none.

Solution: $y = 2$

b. [4 points] Use inequalities to describe the regions in the slope field of the differential equation where the solution curves are increasing.

Solution: The regions in the slope field in which the solution curves are increasing can be determined by finding where

$$\frac{dy}{dx} = (x - y)(y - 2) > 0.$$

Region 1: $x - y > 0$ and $y - 2 > 0$. In other words $x > y$ and $y > 2$.

Region 2: $x - y < 0$ and $y - 2 < 0$, or $x < y$ and $y < 2$

7. [8 points] Consider the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{3n}(x-5)^n.$$

In the following questions, support your answers by stating and properly justifying any test(s), facts and computations you use to prove convergence or divergence. Show all your work.

a. [4 points] Find the radius of convergence of the power series.

Solution:

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{3(n+1)}(x-5)^{n+1}}{\frac{2^n}{3n}(x-5)^n} \right| = |x-5| \lim_{n\to\infty} \frac{2n}{n+1} = 2|x-5| \qquad \text{then} \qquad \frac{1}{R} = 2$$

or

$$R = \lim_{n\to\infty} \left| \frac{\frac{2^n}{3n}}{\frac{2^{n+1}}{3(n+1)}} \right| = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

Radius of convergence=0.5

b. [4 points] Find the interval of convergence of the power series. Make sure to cite all the tests you use to find your answer.

Solution: Testing the endpoints:

•$x = 4.5$: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3n}(4.5-5)^n = \frac{1}{3}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test.

•$x = 5.5$: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3n}(5.5-5)^n = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test $p = 1 \leq 1$.

Interval of convergence: $4.5 \leq x < 5.5$.

8. [12 points]

a. [4 points] Let a be a positive constant. Determine the first three nonzero terms of the Taylor series for

$$f(x) = \frac{1}{(1+ax^2)^4}$$

centered at $x = 0$. Show all your work. Your answer may contain a.

Solution: Using the binomial series $(1+u)^p$ with $u = ax^2$ and $p = -4$

$$f(x) = \frac{1}{(1+ax^2)^4} \approx 1 + pu + \frac{p(p-1)}{2}u^2 = 1 - 4ax^2 + 10a^2x^4$$

b. [2 points] What is the radius of convergence of the Taylor series for $f(x)$? Your answer may contain a.

Solution: Since the interval of convergence of the binomial series is $-1 < u < 1$, then the interval of convergence of the series for $f(x)$ is $-1 < ax^2 < 1$, or $-\sqrt{\frac{1}{a}} < x < \sqrt{\frac{1}{a}}$. Hence the radius of convergence is $R = \sqrt{\frac{1}{a}}$.

c. [3 points] Determine the first three nonzero terms of the Taylor series for

$$g(t) = \int_0^t \frac{1}{(1+ax^2)^4} dx,$$

centered at $t = 0$. Show all your work. Your answer may contain a.

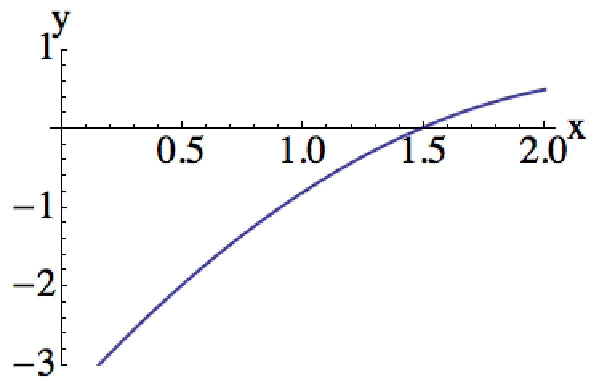
Solution:

$$g(t) = \int_0^t \frac{1}{(1+ax^2)^4} dx \approx \int_0^t 1 - 4ax^2 + 10a^2x^4 dx = x - \frac{4}{3}ax^3 + 2a^2x^5|_0^t = t - \frac{4}{3}at^3 + 2a^2t^5$$

d. [3 points] The degree-2 Taylor polynomial of the function $h(x)$, centered at $x = 1$, is

$$P_2(x) = a + b(x-1) + c(x-1)^2.$$

The following is a graph of $h(x)$:

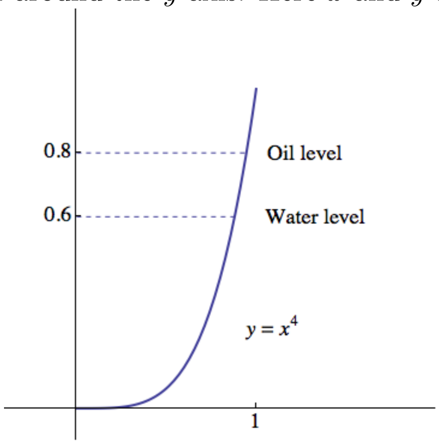


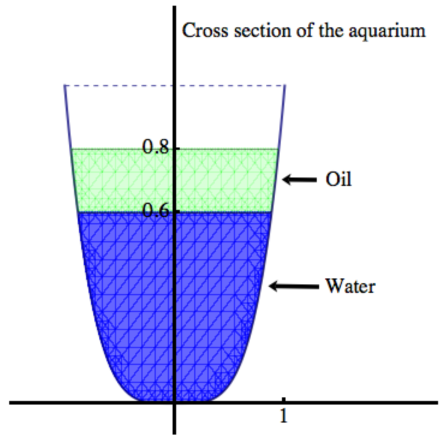
What can you say about the values of a, b, c? You may assume a, b, c are nonzero. Circle your answers. No justification is needed.

Solution:

a is:	Positive	**NEGATIVE**	Not enough information
b is:	**POSITIVE**	Negative	Not enough information
c is:	Positive	**NEGATIVE**	Not enough information

9. [13 points] Olive oil have been poured into the Math Department's starfish aquarium! The shape of the aquarium is a solid of revolution, obtained by rotating the graph of $y = x^4$ for $0 \le x \le 1$ around the y-axis. Here x and y are measured in meters.





The aquarium contains water up to a level of $y = 0.6$ meters. There is a layer of oil of thickness 0.2 meters floating on top of the water. The water and olive oil have densities 1000 and 800 kg per m^3, respectively. Use the value of $g = 9.8$ m per s^2 for the acceleration due to gravity.

a. [6 points] Give an expression involving definite integrals that computes the total mass of the water in the aquarium.

Solution: $\text{Mass}_{water} = \int_0^{0.6} \pi(\sqrt[4]{y})^2(1000)dy = \int_0^{0.6} \pi\sqrt{y}(1000)dy$

b. [7 points] Give an expression involving definite integrals that computes the work necessary to pump all the olive oil to the top of the aquarium.

Solution: $\text{Work}_{oil} = \int_{0.6}^{0.8} \pi(\sqrt[4]{y})^2(800)(9.8)(1-y)dy = \int_{0.6}^{0.8} \pi\sqrt{y}(800)(9.8)(1-y)dy$

10. [13 points] The blockbuster action movie *Mildred's Adventures with Calculus!* was just released. During the first week after the premiere, 2.5 million people went to see it. The studio has conducted a study to gauge the impact of the film on audiences, and found that: *the number of tickets sold in a given week is 60% of the number of tickets sold the previous week.* Assume that this process repeats every week.

a. [5 points] Let p_k be the number of movie tickets, in millions, sold during the kth week after the premiere of the movie. Determine p_2, p_3 and a formula for p_k.

Solution:

$$\begin{aligned} p_1 &= 2.5 \\ p_2 &= 2.5(0.6) \\ p_3 &= 2.5(0.6)^2 \\ p_k &= 2.5(0.6)^{k-1}. \end{aligned}$$

b. [6 points] A movie ticket costs \$8. Let T_n be the total amount of money earned in ticket sales, in millions of dollars, during the first n weeks the movie has been exhibited. Determine T_3 and a closed formula for T_n. Show all your work.

Solution:

$$\begin{aligned} T_1 &= 8(2.5) \\ T_2 &= 8(2.5 + 2.5(0.6)) \\ T_3 &= 8(2.5 + 2.5(0.6) + 2.5(0.6)^2) \\ T_n &= 8(2.5 + 2.5(0.6) + 2.5(0.6)^2 + \cdots + 2.5(0.6)^{n-1}) \\ T_n &= 8(2.5)\frac{1-(0.6)^n}{1-.6} = 50(1-(0.6)^n) \end{aligned}$$

c. [2 points] Determine the value of $\lim_{n\to\infty} T_n$.

Solution: $\lim_{n\to\infty} T_n = \lim_{n\to\infty} 50(1-(0.6)^n) = 50$