Math 116 — First Midterm

October 8, 2014

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.
10. You may use a calculator to evaluate any integral unless specifically instructed otherwise. If you use a calculator to evaluate an integral, write the integral you are evaluating on your exam and indicate that you found the answer with a calculator.

Problem	Points	Score
1	13	
2	15	
3	10	
4	14	
5	12	
6	11	
7	13	
8	12	
Total	100	

1. [13 points] Let $g(x)$ be a differentiable, **odd** function and let $G(x)$ be an anti-derivative of $g(x)$ with $G(2) = 0$. A table of values for $g(x)$ and $G(x)$ is provided below. **Be sure to show all of your work**.

x	0	1	2	3	4
$g(x)$	0	2	3	4	5
$G(x)$	-7	-4	0	5	9

a. [2 points] Write down a formula for $G(x)$ in terms of the function $g(t)$.

$G(x) =$ $\int_2^x g(t)dt$

b. [2 points] Compute $\int_0^1 g(x)dx$.

Solution:

$$\int_0^1 g(x)dx = G(1) - G(0) = -4 - (-7) = 3$$

c. [3 points] Compute $\int_{-4}^2 g(x)dx$.

Solution:

$$\begin{aligned}\int_{-4}^2 g(x)dx &= \int_{-4}^{-2} g(x)dx \\ &= -\int_2^4 g(x)dx \\ &= -(G(4) - G(2)) \\ &= G(2) - G(4) \\ &= -9\end{aligned}$$

d. [3 points] Compute $\int_1^3 xg'(x)dx$.

Solution:

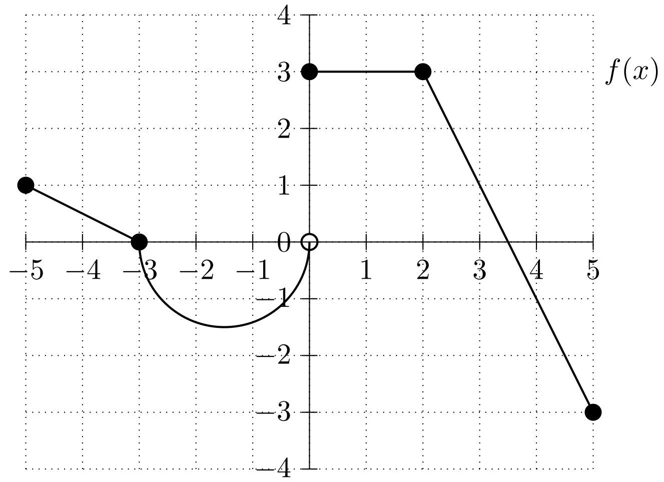
$$\begin{aligned}\int_1^3 xg'(x)dx &= xg(x)|_1^3 - \int_1^3 g(x)dx \\ &= 3g(3) - 1g(1) - (G(3) - G(1)) \\ &= 12 - 2 - (5 - (-4)) \\ &= 1\end{aligned}$$

e. [3 points] Compute $\int_0^1 g(3x)dx$.

Solution:

$$\int_0^1 g(3x)dx = 1/3 \int_0^3 g(x)dx = 1/3(G(3) - G(0)) = 1/3(5 - (-7)) = 4$$

2. [15 points] Below is a graph of the function $f(x)$, comprised of line segments and a semicircle. Let $F(x)$ be an anti-derivative of $f(x)$ with $F(2) = 3$.



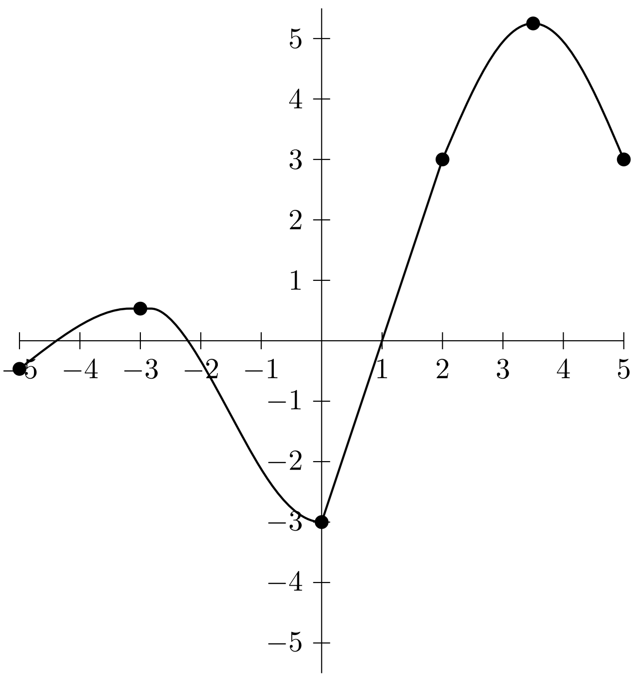
a. [4 points] Find both coordinates of the points where $F(x)$ attains its maximum and minimum values on the interval $-5 \le x \le 5$. No explanation is necessary.

Min:(0 , −3) **Max:** (3.5 , 5.25)

b. [4 points] Find all values of x where $F(x)$ is concave down. Write your answer in the space provided. No explanation is necessary.

$-5 < x < -1.5$ **and** $2 < x < 5$

c. [7 points] Carefully sketch a graph of $F(x)$ on the axes provided below. Be sure to clearly indicate continuity and differentiability in your graph.



3. [10 points] For each of the following compare the two given quantities by writing ">", "<", "=" or "N.I." (for "Not enough information')' in the available answer line. No explanation is necessary.

a. [2 points] Suppose $f(x)$ is continuous and positive.

$$\int_0^1 f(x)dx \quad \underline{\quad > \quad} \quad \int_0^1 xf(x^2)dx$$

b. [2 points] Suppose $\displaystyle\int \frac{1}{(x+2)(x-1)}dx = \int \left(\frac{C}{x+2} + \frac{D}{x-1}\right) dx.$

$$C \quad \underline{\quad < \quad} \quad D$$

c. [2 points] Let $f(x) = x^2$. Let A be the average value of $f(x)$ over the interval $7 \leq x \leq 8$, and let $B = \dfrac{f(13)}{3}$.

$$A \quad \underline{\quad = \quad} \quad B$$

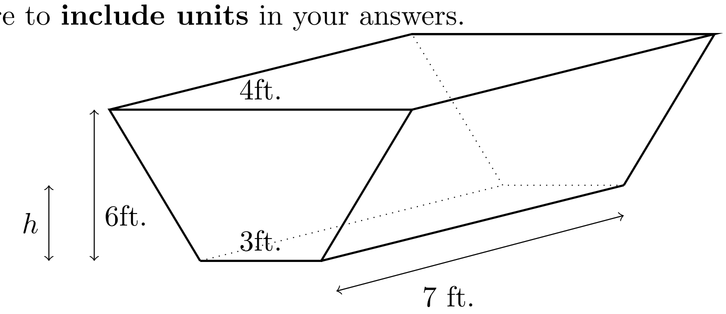
d. [2 points] Let $h(x)$ be a continuous function and let $H(x)$ and $G(x)$ be two anti-derivatives of $h(x)$. Suppose $H(0) > G(0)$.

$$G(1) \quad \underline{\quad < \quad} \quad H(1)$$

e. [2 points] Let $F(x) = \displaystyle\int_0^x f(t)dt$ where $f(t)$ is increasing and positive.

$$F(1) \quad \underline{\quad > \quad} \quad F'(0)$$

4. [14 points] Franklin, your robot, has been digging a ditch in the backyard. The ditch is a trapezoidal prism with length 7 feet, width at the bottom 3 feet and width at the top 4 feet. The ditch descends 6 feet underground. You may assume that each cubic foot of dirt weighs 94 pounds. Be sure to **include units** in your answers.



a. [5 points] Write an expression that approximates the work that Franklin does lifting a slice of dirt Δh feet thick from h feet above the bottom of the ditch, to the top of the ditch.

Solution:

$$\begin{aligned}\text{Work to lift the slice} &= (\text{Displacement})(\text{Weight})\\ &= (6-h)\cdot\delta(7)(\text{Width of the slice})\Delta h\end{aligned}$$

The width of the slice is given by $w = 3 + h/6$

$$\text{Work to lift the slice} = (6-h)\delta(7(3+h/6))\Delta h \quad \text{foot pounds}$$

b. [3 points] Using your answer to part **(a)**, write an integral that computes the total work required to move all of the dirt out over the top of the ditch.

Solution:

$$\text{Total Work} = \int_0^6 (6-h)94(7(3+h/6))dh$$

c. [3 points] To lift the dirt, Franklin uses your old rusty bucket. For Franklin's last bucket-full of dirt, the bucket starts with 30 pounds of dirt and loses 10 pounds of dirt at a constant rate over the 6 feet that it travels vertically. When the bucket is x feet above the bottom of the ditch, what is the weight of the dirt in the bucket?

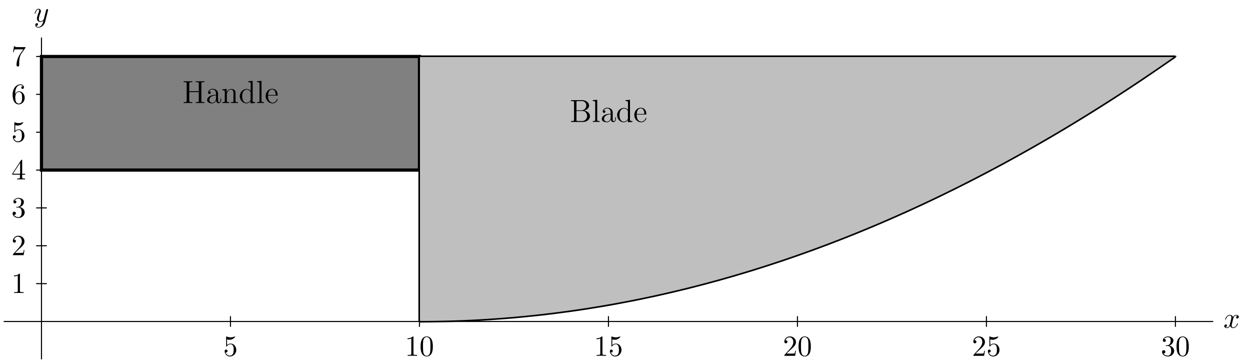
Solution: When the bucket is x feet above the bottom of the hole, it contains $30 - \frac{10x}{6}$ pounds of dirt.

d. [3 points] Using your answer from part **(c)**, find the work required to lift the bucket-full of dirt from the bottom of the ditch to the top.

Solution:

$$\text{Total Work} = \int_0^6 (30 - \frac{10x}{6})dx = 150 \quad \text{foot pounds}$$

5. [12 points] Franklin, your robot, goes to the local store and buys a new chef's knife. The handle of the knife is given by the region contained between the lines $y = 7$, $y = 4$, $x = 0$ and $x = 10$. The blade of the knife is in the shape of the region bounded by the line $x = 10$, $y = 7$ and the curve $y = \frac{7(x-10)^2}{400}$. Assume all lengths are in centimeters. Below is a diagram of the knife.



Assume that the density of the knife is constant, with value δ kg/cm^2.

a. [2 points] Find the total mass of the **handle of the knife**. Include units.

Solution:

$$\text{Mass of handle} = 3 \cdot 10 \cdot \delta = 30\delta \quad \text{kilograms}$$

b. [4 points] Write an expression involving integrals that gives the total mass of the **blade of the knife**. Do not evaluate any integrals.

Solution:

$$\text{Mass of blade} = \int_{10}^{30} \delta\left(7 - \frac{7(x-10)^2}{400}\right)dx = \frac{280}{3}\delta \quad \text{kilograms}$$

c. [2 points] Write an expression involving integrals that gives the x-coordinate of the center of mass of the **blade portion of the knife**. Do not evaluate any integrals.

Solution:

$$\bar{x} = \frac{\int_{10}^{30} x\left(7 - \frac{7(x-10)^2}{400}\right)dx}{\int_{10}^{30} 7 - \frac{7(x-10)^2}{400}dx}$$

d. [4 points] Write an expression involving integrals that gives the x-coordinate of the center of mass of the **whole knife** (the blade and handle together). Do not evaluate any integrals.

Solution:

$$\bar{x} = \frac{\int_0^{10} 3x dx + \int_{10}^{30} x\left(7 - \frac{7(x-10)^2}{400}\right)dx}{30 + \int_{10}^{30} 7 - \frac{7(x-10)^2}{400}dx}$$

6. [11 points] Franklin, your robot, is zipping around the kitchen making his famous "Definitely Not Poison!" soup. His coordinates in the xy-plane are given by the parametric equations

$$x = t^2 - t \qquad y = -\sin(\pi t)$$

t seconds after he starts making soup. Assume that both x and y are measured in meters.

a. [2 points] Calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = \underline{\quad 2t - 1 \quad} \qquad \frac{dy}{dt} = \underline{\quad -\pi\cos(\pi t) \quad}$$

b. [2 points] Find all times t when Franklin's velocity is zero.

Solution: Franklin comes to a stop at all times t when both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

- $\frac{dx}{dt} = 2t - 1 = 0$ when $t = 1/2$.
- $\frac{dy}{dt} = -\pi\cos(\pi t) = 0$ when $t = 1/2, 3/2, 5/2$, etc.

So Franklin comes to a stop when $t = 1/2$.

$t =$ ____ $1/2$ ____

c. [3 points] Find Franklin's **speed** when $t = 2$ seconds. Include units.

Solution:

$$\text{Franklin's speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

When $t = 2$:

- $\frac{dx}{dt} = 2(2) - 1 = 3$
- $\frac{dy}{dt} = -\pi\cos(2\pi) = -\pi$

Franklin's speed when $t = 2$ is $\sqrt{3^2 + \pi^2} \approx 4.34$ meters per second.

Franklin's speed= ____ $\sqrt{3^2 + \pi^2} \approx 4.34$ **m/s** ____

d. [4 points] Write an integral which gives the distance traveled by Franklin during his first five seconds of zipping around. Do not evaluate this integral.

Solution:

$$\int_0^5 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^5 \sqrt{(2t-1)^2 + (-\pi\cos(\pi t))^2}\,dt$$

7. [13 points] Kazilla is designing a new board game. She is interested in using the region R in the xy-plane bounded by $y = 2$, $y = x$, $x = 1$ and $x = 0$.

a. [4 points] The first part of the game is a spinning top formed by rotating the region R around the y-axis. Write an integral (or a sum of integrals) that gives the volume of the spinning top. Do not evaluate your integral(s).

Solution: Shell method:

$$\int_0^1 2\pi(2-x)x\,dx$$

Washer method:

$$\int_0^1 \pi y^2 dy + \int_1^2 \pi dy$$

b. [4 points] Another game piece has a base in the shape R, but with semicircular cross sections **perpendicular** to the x-axis. Write an integral which gives the volume of the game piece. Do not evaluate your integral.

Solution:

$$\frac{\pi}{8}\int_0^1 (2-x)^2 dx$$

c. [5 points] A third game piece has volume given by $\int_0^2 \pi(h(x))^2 dx$ where $h(x)$ is a continuous function of x. Use MID(3) to approximate the volume of this third game piece. Be sure to write out all of the terms in your approximation. Your answer may contain the function $h(x)$.

Solution:

$$MID(3) = \frac{2}{3}[\pi(h(1/3))^2 + \pi(h(1))^2 + \pi(h(5/3))^2]$$

8. [12 points] Circle **all** that apply. You do not need to provide an explanation for your answer.

a. [3 points] Which of the following are parameterizations for the circle of radius 2 centered at the origin for $0 \le t \le \pi$?

- ○ $x = \cos(2t), \quad y = \sin(2t)$
- ○ $x = \sqrt{2}\cos(t), \quad y = \sqrt{2}\sin(t)$
- ○ $\boxed{x = 2\cos(2t), \quad y = -2\sin(2t)}$

b. [3 points] Which of the following are antiderivatives of e^{-x^4}?

- ○ $\int_1^{x^4} e^{-t}dt$
- ○ $\boxed{\int_1^{x} e^{-t^4}dt}$
- ○ $\boxed{-\frac{1}{2}\int_{2x}^{1} e^{-t^4/16}dt}$

c. [3 points] Suppose that $F(x)$ is an odd function which is concave down and increasing for $x \ge 0$. Which of the following approximations is certain to give an overestimate for $\int_{-2}^{1} F(x)dx$?

- ○ LEFT(4)
- ○ $\boxed{\text{RIGHT}(4)}$
- ○ $\boxed{\text{TRAP}(3)}$

d. [3 points] The integral $\int_0^1 4\pi z^2(1-z^2)dz$ gives the volume of which of the following solids of revolution?

- ○ $\boxed{\text{The region bounded by } y = 2x\sqrt{1-x^2} \text{ and } y = 0 \text{ rotated about the } x\text{-axis.}}$
- ○ $\boxed{\text{The region bounded by } y = 0,\ x = 1 \text{ and } y = 2x^2(1+x) \text{ rotated about line } x = 1.}$
- ○ $\boxed{\text{The region bounded by } x = 2y \text{ and } x = 2y^2 \text{ rotated about the } y\text{-axis.}}$