Math 116 — Second Midterm

November 12, 2014

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

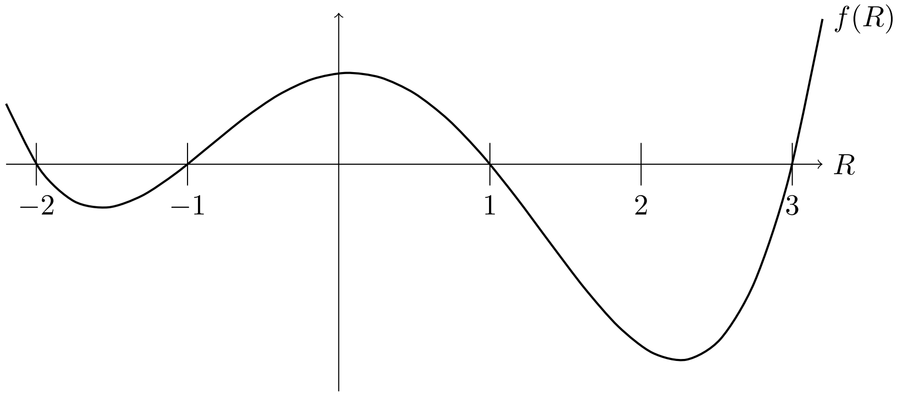
1. **Do not open this exam until you are told to do so.**
2. This exam has 13 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.
10. You may use a calculator to evaluate any integral unless specifically instructed otherwise. If you use a calculator to evaluate an integral, write the integral you are evaluating on your exam and indicate that you found the answer with a calculator.

Problem	Points	Score
1	12	
2	11	
3	16	
4	12	
5	10	
6	10	
7	8	
8	11	
9	10	
Total	100	

1. [12 points] Franklin, your robot, is on the local news. Let $R(t)$ be the number of robots, in millions, that have joined the robot uprising t minutes after the start of the broadcast. After watching the news for a little bit, you find that $R(t)$ obeys the differential equation:

$$\frac{dR}{dt} = f(R)$$

for some function $f(R)$. A graph of $f(R)$ is shown below.



a. [3 points] If $R(t)$ is the solution to the above differential equation with $R(0) = 0$, what is $\lim_{t\to\infty} R(t)$? Justify your answer.

Solution: If $R = 0$, $f(R) = R'(t)$ is positive, so R will increase as t increases. As R increases to 1, $R'(t) = f(R)$ goes to 0, so $\lim_{t\to\infty} R(t) = 1$.

b. [6 points] Find the equilibrium solutions to the above differential equation **and** classify them as stable or unstable.

Solution:

$R = -2$	$\boxed{\textbf{Stable}}$	**Unstable**
$R = -1$	**Stable**	$\boxed{\textbf{Unstable}}$
$R = 1$	$\boxed{\textbf{Stable}}$	**Unstable**
$R = 3$	**Stable**	$\boxed{\textbf{Unstable}}$

c. [3 points] Let $R(t)$ be a solution to the given differential equation, with $R(3) = 0.5$. Is the graph of $R(t)$ concave up, concave down, or neither at the point $(3, 0.5)$? Justify your answer.

Solution:

$$\frac{d^2R}{dt^2} = \frac{d}{dt}f(R) = f'(R)\frac{dR}{dt} = f'(R)f(R)$$

At $R = 0.5$, $f'(0.5) < 0$ and $f(0.5) > 0$ so $\frac{d^2R}{dt^2} < 0$. Therefore, the solution curve will be concave down.

2. [11 points] Kazilla sends you on a very important trip to the store. Rather than give you directions, she provides you with the differential equation

$$\frac{dy}{dx} = x + xy + 1$$

which gives $x-$ and $y-$coordinates on your map. Your current position on the map is the point $(0, -1)$.

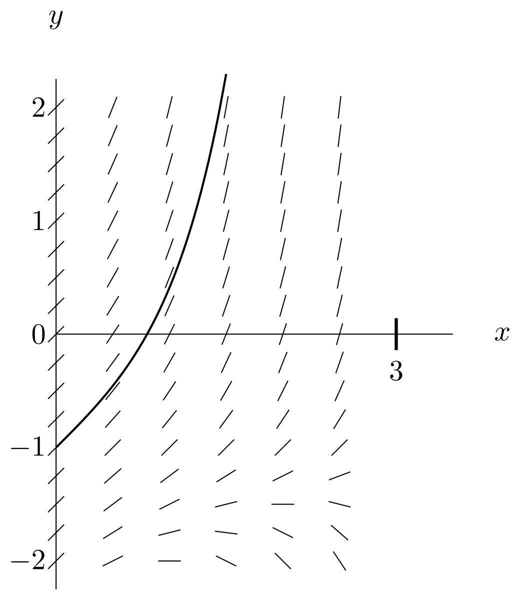
a. [1 point] Is this differential equation separable? Circle one. **Yes** $\boxed{\textbf{No}}$

b. [5 points] Kazilla tells you to follow the solution curve to the differential equation from your current position to $x = 1.5$ to find the location of the store on the map. Use Euler's method with step size $\Delta x = 0.5$ to approximate the $y-$coordinate of the store.

Solution:

x	y	$\Delta y = \Delta x(x + xy + 1)$
0	-1	$0.5(0 + 0(-1) + 1) = 0.5$
0.5	-0.5	$0.5(0.5 + 0.5(-0.5) + 1) = 0.625$
1	0.125	$0.5(1 + (1(0.125) + 1) = 1.0625$
1.5	1.1875	-

c. [2 points] The slope field of the differential equation is given below. Sketch the solution passing through $(0, -1)$. The point (3,0) is labeled for scale.



d. [3 points] Is the estimate for the position of the store you found in part **(b)** above or below the actual position of the store on your map? Justify your answer.

Solution: Euler's method uses tangent line approximations. In part **(c.)** we saw that the actual solution curve is concave up over the region $0 \le x \le 1.5$, so Euler's method will give an underestimate for the actual value. Therefore our estimate in part **(b).** is be low the actual position of the store.

3. [16 points]

a. [5 points] Kazilla uses the ingredients you acquired from the store to make a special potion. Kazilla starts with a bucket containing 5 L of water and steadily adds purple liquid to the bucket at a rate of 0.5 L/min. Once in the bucket, the purple liquid mixes instantly with the water and the mixture drains out of the bottom of the bucket at a rate of 0.5 L/min. Let $P(t)$ give the amount of purple liquid in the bucket t minutes after Kazilla starts making the potion. Write a differential equation involving $P(t)$.

Solution:

$$\frac{dP}{dt} = 0.5 - \frac{P}{10}$$

b. [7 points] Later, Kazilla asks you to find the correct amount of green liquid to add to her potion. Let $G(t)$ be the total amount of green liquid you need to add, in liters, after t minutes. Suppose $G(1) = \frac{3}{2}$ and $G(t)$ satisfies the differential equation

$$\frac{dG}{dt} = -2t(G-1)^2.$$

Find $G(t)$.

Solution: Start with a separation of variables:

$$\int \frac{dG}{(G-1)^2} = \int -2t dt$$

$$\frac{-1}{(G-1)} = -t^2 + C$$

$$G = \frac{1}{t^2 + C} + 1$$

Using the initial condition that $G(1) = 3/2$, we can solve for C:

$$\frac{3}{2} = \frac{1}{1+C} + 1$$

$$C = 1$$

So $G(t) = \dfrac{1}{t^2+1} + 1$

c. [4 points] Finally, Kazilla asks you to add an amount of blue liquid $B(t)$ at a rate given by

$$\frac{dB}{dt} = -2B + e^{t/2}.$$

For which value(s) of c is the function

$$B(t) = 7e^{-2t} + \frac{ce^{t/2}}{5}$$

a solution to this differential equation?

Solution:

$$B'(t) = -14e^{-2t} + \frac{ce^{t/2}}{10}$$

$$B'(t) = \frac{dB}{dt} = -2(7e^{-2t} + \frac{ce^{t/2}}{5}) + e^{t/2} = -14e^{-2t} + (1 - \frac{2c}{5})e^{t/2}$$

So $1 - \frac{2c}{5} = \frac{c}{10}$ or $c = 2$.

4. [12 points] Consider the following sequences:

$$f_n = \frac{\pi^n}{e^n} \qquad g_n = (-1)^n \sin(n) \qquad h_n = \cos(e^{-n}) \qquad i_n = \int_1^n \frac{1}{(x+3)^2}dx$$

For each sequence, circle **all** that apply. No justification is necessary.

a. [2 points] The sequence (f_n) is :

Bounded $\boxed{\text{Increasing}}$ Decreasing

b. [2 points] The sequence (g_n) is :

$\boxed{\text{Bounded}}$ Increasing Decreasing

c. [2 points] The sequence (h_n) is :

$\boxed{\text{Bounded}}$ $\boxed{\text{Increasing}}$ Decreasing

d. [2 points] The sequence (i_n) is :

$\boxed{\text{Bounded}}$ $\boxed{\text{Increasing}}$ Decreasing

e. [4 points] For each given sequence, if it converges, determine its limit and write that limit in the space provided. If the sequence diverges, write "diverges". No justification is necessary.

(f_n): **diverges** (h_n): **1**

(g_n): **diverges** (i_n): **1/4**

5. [10 points] Determine whether the following integrals converge or diverge. If the integral converges, find the exact value. You must show all work, and perform any integral computations by hand.

a. [5 points] $\displaystyle\int_2^\infty \frac{dx}{x(\ln(x))^2}$

Solution: We start with the substitution $u = \ln(x)$:

$$\int_2^\infty \frac{dx}{x(\ln(x))^2} = \int_{\ln(2)}^\infty \frac{du}{u^2}$$

Now we integrate normally:

$$\begin{aligned}\int_{\ln(2)}^\infty \frac{du}{u^2} &= \lim_{a\to\infty} \int_{\ln(2)}^a \frac{du}{u^2} \\ &= \lim_{a\to\infty} -\frac{1}{u}\Big|_{\ln(2)}^a \\ &= \lim_{a\to\infty} -\frac{1}{a} + \frac{1}{\ln(2)} \\ &= \frac{1}{\ln(2)}\end{aligned}$$

b. [5 points] $\displaystyle\int_1^3 \frac{x}{(x-3)^3}dx$

Solution: You can do this problem with integration by parts and by substitution. For substitution, use $u = (x-3)$, so the integral becomes:

$$\int_1^3 \frac{x}{(x-3)^3}dx = \int_{-2}^0 \frac{u+3}{u^3}du$$

This integral splits up as:

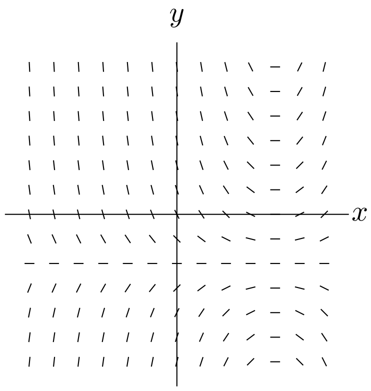
$$\begin{aligned}
\int_{-2}^0 \frac{3}{u^3} + \frac{1}{u^2}du &= \lim_{a\to 0} \int_{-2}^a \frac{3}{u^3} + \frac{1}{u^2}du \\
&= \lim_{a\to 0^-} \left(\frac{-3}{2u^2} - \frac{1}{u}\right)\Big|_{-2}^a \\
&= \lim_{a\to 0^-} \left(\frac{-3/2 - u}{u^2}\right)\Big|_{-2}^a \\
&= \lim_{a\to 0^-} \left(\frac{-3/2 - a}{a^2}\right) - \left(\frac{-3/2-(-2)}{(-2)^2}\right)
\end{aligned}$$

But $\displaystyle\lim_{a\to 0^-} \left(\frac{-3/2-a}{a^2}\right) = -\infty$, so the integral diverges. To solve this by parts, set $u = x$ and $dv = \frac{1}{(x-3)^3}$, so $du = dx$ and $v = \frac{-1}{(x-3)^2}$.

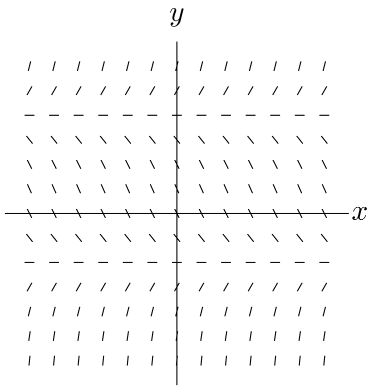
$$\begin{aligned}
\int_1^3 \frac{x}{(x-3)^3}dx &= \lim_{a\to 3^-} \left(\frac{-x}{(x-3)^2}\Big|_1^a - \int_1^a \frac{-1}{(x-3)^2} \right) \\
&= \lim_{a\to 3^-} \left(\frac{-x-(x-3)}{(x-3)^2}\Big|_1^a \right) \\
&= -\infty
\end{aligned}$$

6. [10 points] Match the following. For each blank, there is only one correct answer.

a. [4 points] For each slope field on the left, write the letter corresponding to the differential equation that generates that slope field in the blank provided.



I. *(E.)*



II. *(B.)*

(A.) $\dfrac{dy}{dx} = (y+2)(y-1)$

(B.) $\dfrac{dy}{dx} = (y-2)(y+1)$

(C.) $\dfrac{dy}{dx} = (y+1)(y-2)^2$

(D.) $\dfrac{dy}{dx} = (2-x)(y+1)$

(E.) $\dfrac{dy}{dx} = (x-2)(y+1)$

(F.) $\dfrac{dy}{dx} = (x-1)(y-2)$

b. [6 points] Let $r(\theta) = k$ be a polar curve where $k > 0$ is a constant. Match the quantities on the left with their formulas (in terms of θ) on the right.

I. $\dfrac{dy}{d\theta} =$ *(A.)*

II. $\dfrac{dx}{d\theta} =$ *(D.)*

III. $\dfrac{dy}{dx} =$ *(H.)*

(A.) $k\cos(\theta)$

(B.) $-k\cos(\theta)$

(C.) $k\sin(\theta)$

(D.) $-k\sin(\theta)$

(E.) $\tan(\theta)$

(F.) $-\tan(\theta)$

(G.) $\frac{1}{\tan(\theta)}$

(H.) $-\frac{1}{\tan(\theta)}$

7. [8 points] Suppose that $f(x)$ is a differentiable function, defined for $x > 0$, which satisfies the inequalities $0 \leq f(x) \leq \frac{1}{x}$ for $x > 0$. Determine whether the following statements are always, sometimes or never true by circling the appropriate answer. No justification is necessary.

a. [2 points] $\displaystyle\int_1^\infty f(x)dx$ converges.

Always $\boxed{\textbf{Sometimes}}$ **Never**

b. [2 points] $\displaystyle\int_1^\infty (f(x))^2 dx$ converges.

$\boxed{\textbf{Always}}$ **Sometimes** **Never**

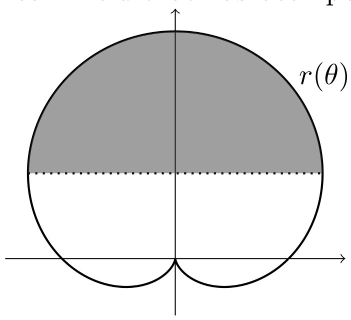
c. [2 points] $\displaystyle\int_0^1 f(x)dx$ converges.

Always $\boxed{\textbf{Sometimes}}$ **Never**

d. [2 points] $\displaystyle\int_1^\infty e^{f(x)}dx$ converges.

Always **Sometimes** $\boxed{\textbf{Never}}$

8. [11 points] Franklin, your robot, uses the lasers on his satellites to burn strange shapes in local corn fields. One of these strange shapes is given by the polar equation $r(\theta) = 2 + 2\sin(\theta)$ where $r(\theta)$ is measured in kilometers. All of the corn **above** the line $y = \frac{3}{2}$ has been pecked away by a flock of wild chickens. The polar curve $r(\theta)$ (solid) and the line $y = \frac{3}{2}$ (dotted) are shown below. The portion of the corn field that has been pecked away is shaded below.



a. [6 points] Write an expression involving one or more integrals which gives the area of the shaded region. Do not evaluate any integrals. **Include units.**

Solution: First, we need to find the values of θ where $r(\theta)$ intersects the line $y = 3/2$. Since $y = r\sin(\theta)$, we have that

$$3/2 = [2 + 2\sin(\theta)]\sin(\theta)$$

This is a quadratic equation in $\sin(\theta)$, and the quadratic formula gives $\sin(\theta) = 1/2$ or $-3/2$. Since $\sin(\theta)$ is bounded between -1 and 1, $\sin(\theta) = 1/2$. This means that:

$$\theta = \pi/6 \qquad \text{or} \qquad 5\pi/6$$

The line $y = 3/2$ has polar equation $r_{\text{line}}(\theta) = \frac{3}{2\sin(\theta)}$. Therefore the area bounded between $r(\theta)$ and $r_{\text{line}}(\theta)$ is given by the integral:

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} r(\theta)^2 - r_{\text{line}}(\theta)^2 d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (2 + 2\sin(\theta))^2 - \left(\frac{3}{2\sin(\theta)}\right)^2 d\theta \qquad \text{km}^2$$

b. [5 points] Write an expression involving one or more integrals which gives the perimeter of the shaded region. Do not evaluate any integrals. **Include units.**

Solution: To find the length of the portion of the line $y = 3/2$ passing through the shaded region, we can take the difference of the x-coordinates of the points where $r(\theta)$ intersects the line. Since $r_{\text{line}}(\theta) = \frac{3}{2\sin(\theta)}$, the x-coordinates are given by $x = r_{\text{line}}(\theta)\cos(\theta) = \frac{3\cos(\theta)}{2\sin(\theta)}$

$$\text{Length of segment} = \frac{3\cos(\pi/6)}{2\sin(\pi/6)} - \frac{3\cos(5\pi/6)}{2\sin(5\pi/6)} \approx 5.196 \qquad \text{km}$$

Now we need to calculate the portion of the perimeter lying on $r(\theta)$. For this we can use the polar perimeter formula

$$\text{Length over } r(\theta) = \int_{\pi/6}^{5\pi/6} \sqrt{(2+2sin(\theta))^2 + (2\cos(\theta))^2}d\theta \quad \text{km}$$

$$\text{Perimeter} = \frac{3\cos(\pi/6)}{2\sin(\pi/6)} - \frac{3\cos(5\pi/6)}{2\sin(5\pi/6)} + \int_{\pi/6}^{5\pi/6} \sqrt{(2+2sin(\theta))^2 + (2\cos(\theta))^2}d\theta \quad \text{km}$$

9. [10 points] Determine whether the following integrals converge or diverge. Justify your answer.

a. [5 points] $\displaystyle\int_1^\infty \frac{x^2}{e^{-x}+3x^3}dx$

Solution:
We know that $e^{-x} < x^3$ for $x > 1$. Therefore

$$\int_1^\infty \frac{x^2}{e^{-x}+3x^3}dx > \int_1^\infty \frac{x^2}{4x^3}dx$$

We can simplify the integral on the right:

$$\int_1^\infty \frac{x^2}{4x^3}dx = \frac{1}{4}\int_1^\infty \frac{1}{x}dx$$

We know that the integral $\displaystyle\int_1^\infty \frac{1}{x}dx$ diverges, so this larger integral must also diverge.

b. [5 points] $\displaystyle\int_0^1 \frac{x\cos(x)-\sin(x)}{x^2}dx$

$\left(\textit{Hint}: \quad \frac{d}{dx}\left(\frac{\sin(x)}{x}\right) = \frac{x\cos(x)-\sin(x)}{x^2}.\right)$

Solution:

$$\begin{aligned}\int_0^1 \frac{x\cos(x)-\sin(x)}{x^2}dx &= \lim_{b\to 0^+}\int_b^1 \frac{x\cos(x)-\sin(x)}{x^2}dx\\ &= \lim_{b\to 0^+}\left.\frac{\sin(x)}{x}\right|_{x=b}^1\\ &= \frac{\sin(1)}{1} - \lim_{b\to 0^+}\frac{\sin(b)}{b}\end{aligned}$$

To evaluate the limit, we need to use L'Hopital's Rule:

$$\lim_{b\to 0}\frac{\sin(b)}{b} = \lim_{b\to 0}\frac{\cos(b)}{1} = 1.$$

We can now give an exact value for the integral.

$$\int_0^1 \frac{x\cos(x)-\sin(x)}{x^2}dx = \frac{\sin(1)}{1} - 1 < \infty$$

and therefore the integral converges.