Math 116 — Final Exam

December 12, 2014

Name: EXAM SOLUTIONS

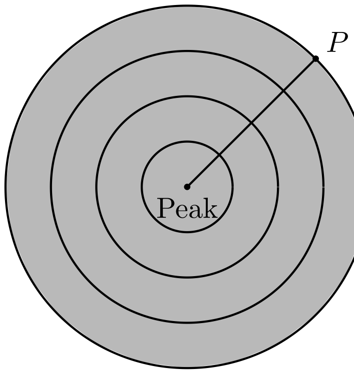
Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 14 pages including this cover. There are 12 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. On the last page of this exam you will find a page containing formulas for some common Taylor series.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	6	
3	3	
4	8	
5	11	
6	8	
7	14	
8	7	
9	5	
10	10	
11	9	
12	8	
Total	100	

1. [11 points] You are standing on Lonely Hill while Franklin's robot army surrounds you. You have an Electro-Magnetic Pulse (EMP) device that deactivates all of the robots within a one mile radius of the peak of Lonely Hill as shown in the shaded region below. The point P is one mile from Lonely Hill for your reference. Let $R(x)$ give the density, in robots per square mile, of robots x miles from the peak of Lonely Hill. A table of values for $R(x)$ is given below.

x	0	0.25	0.5	0.75	1
$R(x)$	50	100	40	35	15



a. [3 points] Write an integral which gives the exact number of robots that are deactivated by the EMP. Your answer may contain the function $R(x)$. Do not evaluate your integral.

Solution:

$$2\pi \int_0^1 xR(x)dx$$

b. [5 points] Use the Trapezoidal method with as many subdivisions as possible to estimate the total number of robots that are deactivated by the EMP. Write out all of the terms for your estimate.

Solution:

$$\begin{aligned} LEFT(4) &= 2\pi \cdot 0.25((0)50 + (0.25)100 + (0.5)40 + (0.75)35) \\ RIGHT(4) &= 2\pi \cdot 0.25((0.25)100 + (0.5)40 + (0.75)35 + (1)15) \\ TRAP(4) &= \frac{RIGHT(4) + LEFT(4)}{2} \approx 123.7 \end{aligned}$$

c. [3 points] You are very happy that the EMP worked. Let $H(t)$ give the level of happiness you feel t seconds after Franklin's robots are deactivated. Determine the value of a so that $H(t) = 3e^t + at^2 - 4t - 4$ is a solution to the differential equation

$$\frac{dH}{dt} = H + 2t^2.$$

Solution:

$$H'(t) = 3e^t + 2at - 4$$

$$\begin{aligned} 3e^t + 2at - 4 &= (3e^t + at^2 - 4t - 4) + 2t^2 \\ 2at &= at^2 - 4t + 2t^2 \\ 2a &= at - 4 + 2t \\ a(2-t) &= -2(2-t) \\ a &= -2 \end{aligned}$$

This implies that $at^2 + t^2 = 0$ and $2at = -4t$, so $a = -2$.

2. [6 points] Let $f(x) = xe^{-x^2}$.

a. [4 points] Find the Taylor series of $f(x)$ centered at $x = 0$. Be sure to include the first 3 nonzero terms and the general term.

Solution: We can use the Taylor series of e^y to find the Taylor series for e^{-x^2} by substituting $y = -x^2$.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots$$

Therefore the Taylor series of xe^{-x^2} is

$$xe^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!} = x - x^3 + \frac{x^5}{2!} + \cdots + \frac{(-1)^n x^{2n+1}}{n!} + \cdots$$

b. [2 points] Find $f^{(15)}(0)$.

Solution: We know that $\frac{f^{(15)}(0)}{15!}$ will appear as the coefficient of the degree 15 term of the Taylor series. Using part (**a**), we see that the degree 15 term has coefficient $\frac{-1}{7!}$. Therefore

$$f^{(15)}(0) = \frac{-15!}{7!} = -259,459,200$$

3. [3 points] Determine the exact value of the infinite series

$$1 - \frac{2}{1!} + \frac{4}{2!} - \frac{8}{3!} + \cdots + \frac{(-1)^n 2^n}{n!} + \cdots$$

Solution: Notice that this is the Taylor series for e^y applied to $y = -2$. Therefore, the series has exact value e^{-2}.

4. [8 points] Franklin's robots start building more robots to replace their deactivated comrades. The initial number of robots in Franklin's army is 800. Each minute, the number of robots increases by 15%. At the end of each minute, you fire an EMP which immediately deactivates 50 robots.

a. [3 points] Let R_n denote the number of active robots in Franklin's army immediately after the EMP is fired for the n-th time. Find R_1 and R_2.

Solution:

$$R_1 = (1.15)800 - 50$$
$$R_2 = (1.15)((1.15)800 - 50) - 50$$

b. [4 points] Find a closed form expression for R_n (i.e. evaluate any sums and solve any recursion).

Solution:

$$\begin{aligned} R_n &= 800(1.15)^n - \sum_{i=0}^{n-1} 50(1.15)^i \\ &= 800(1.15)^n - \frac{50(1 - 1.15^n)}{1 - 1.15} \end{aligned}$$

c. [1 point] Find $\lim_{n\to\infty} R_n$. No justification is necessary.

Solution:

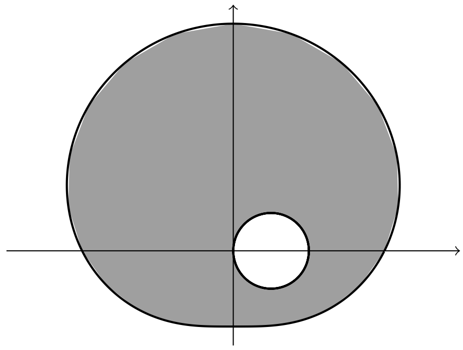
$$\lim_{n\to\infty} R_n = \infty$$

5. [11 points] Franklin's robot army is surrounding you!

a. [6 points] Consider the polar curves

$$r = \cos(\theta) \qquad\qquad r = \sin(\theta) + 2$$

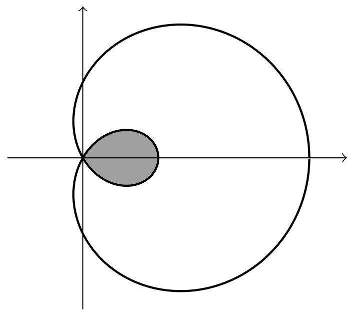
Franklin's robot army occupies the shaded region between these two curves. Write an expression involving integrals that gives the **area** occupied by Franklin's robot army. Do not evaluate any integrals.



Solution:

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (\sin(\theta)+2)^2 d\theta - \frac{1}{2}\int_0^{\pi} (\cos(\theta))^2 d\theta$$

b. [5 points] Your friend, Kazilla, pours her magic potion on the ground. Suddenly, a flock of wild chickens surrounds you. The chickens occupy the shaded region enclosed within the polar curve $r = 1 + 2\cos(\theta)$ as shown below. Write an expression involving integrals that gives the **perimeter** of the region occupied by the flock of wild chickens. Do not evaluate any integrals.



Solution: We use the arc length formula:

$$\text{Arc Length} = \int_a^b \sqrt{(r(\theta))^2 + (r'(\theta))^2} d\theta$$

Note that $r'(\theta) = -2\sin(\theta)$. Also, the shaded region of lies between $\theta = 2\pi/3$ and $\theta = 4\pi/3$ (you can see this by setting $r(\theta) = 0$, and testing that $r(\pi) = -1$, so it lies on the boundary of the shaded region.)

$$\text{Arc Length} = \int_{2\pi/3}^{4\pi/3} \sqrt{(1+2\cos(\theta))^2 + (-2\sin(\theta))^2} d\theta$$

6. [8 points] Suppose that $f(x)$, $g(x)$, $h(x)$ and $k(x)$ are all positive, differentiable functions. Suppose that

$$0 < f(x) < \frac{1}{x} < g(x) < \frac{1}{x^2}$$

for all $0 < x < 1$, and that

$$0 < h(x) < \frac{1}{x^2} < k(x) < \frac{1}{x}$$

for $x > 1$. Determine whether the following statements are always, sometimes or never true by circling the appropriate answer. No justification is necessary.

a. [2 points] $\int_0^1 g(x)dx$ converges.

Always **Sometimes** $\boxed{\textbf{Never}}$

b. [2 points] $\int_0^1 f(x)dx$ diverges.

Always $\boxed{\textbf{Sometimes}}$ **Never**

c. [2 points] $\sum_{n=1}^{\infty} h(n)$ diverges.

Always **Sometimes** $\boxed{\textbf{Never}}$

d. [2 points] $\sum_{n=1}^{\infty} k(n)$ converges.

Always $\boxed{\textbf{Sometimes}}$ **Never**

7. [14 points] Chickens continue to appear around you, and Franklin's army is hesitant to advance.

a. [6 points] Let $F(t)$ give the total number of chickens that have arrived after t seconds. You observe that $F(t)$ obeys the following differential equation

$$\frac{dF}{dt} = e^{-F}t^2.$$

If there are initially 20 chickens, find a formula (in terms of t) for $F(t)$.

Solution:

$$\int e^F df = \int t^2 dt$$
$$e^F = \frac{t^3}{3} + C$$
$$F(t) = \ln(\frac{t^3}{3} + C)$$

Since $F(0) = 20$, we see that

$$20 = \ln(C)$$

so $C = e^{20}$, and

$$F(t) = \ln(\frac{t^3}{3} + e^{20})$$

b. [4 points] A large, familiar-looking chicken steps forward from the flock and clucks, "Koo Koo Katcha!". This large chicken waddles towards Franklin following the parametric equations

$$x(t) = \frac{\sin(\pi t) + 1}{\pi} \qquad y(t) = \ln(t + 1)$$

where t is the time, in seconds, after the chicken steps forward from the flock and both x and y are measured in feet. Find the chicken's speed 10 seconds after it steps forward. Include units.

Solution:

$$x'(t) = \cos(\pi t) \qquad y'(t) = \frac{1}{t+1}$$

Now we plug these into the speed formula

$$\text{Speed} = \sqrt{(x'(t))^2 + (y'(t))^2)}$$

when $t = 10$.

$$\text{Speed} = \sqrt{(\cos(10\pi))^2 + (\frac{1}{11})^2} = \frac{\sqrt{122}}{11}$$

c. [4 points] Franklin says, "`BEEP BOOP BEEP. YOU'RE RIGHT, WHAT HAVE I BECOME?`" A single robot tear falls from Franklin's robot eye. Consider the region in the xy-plane bounded by $y = \dfrac{\sin(x)}{x+2}$, $x = \pi$, $x = 2\pi$, and the x-axis. The volume of Franklin's tear is given by rotating this region around the x-axis. Write an integral giving the volume of Franklin's tear. Do not evaluate this integral.

Solution:

$$\int_{\pi}^{2\pi} \pi \left(\frac{\sin(x)}{x+2}\right)^2 dx$$

8. [7 points] Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n n}$.

a. [2 points] At which x-value is the interval of convergence of this power series centered?

Solution: This power series is centered on $x = -2$.

b. [5 points] The radius of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n n}$ is 3. Find the interval of convergence for this power series. Thoroughly justify your answer.

Solution: Since the radius of convergence for this power series is 3 and it is centered on $x = 2$, the interval of convergence contains the open interval $(-2-3, -2+3) = (-5, 1)$. Now we only need to check the endpoints $x = -5$ and $x = 1$.

- For $x = 1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(1+2)^n}{3^n n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test with $p = 1$ (this is the harmonic series).
- For $x = -5$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-5+2)^n}{3^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the alternating series test.

Therefore, the interval of convergence for this power series is $[-5, 1)$.

9. [5 points] Find the radius of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^{2n}$

Solution: Let the n-th term be denoted by a_n

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \left| \frac{(2(n+1))! x^{2(n+1)}}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)! x^{2n}} \right| \\ &= \left| \frac{(2n+2)(2n+1)x^2}{((n+1)^2} \right| \end{aligned}$$

Therefore, we can use the ratio test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)x^2}{((n+1)^2} \right| = 4x^2.$$

So this series converges for x with $4x^2 < 1$, or rather with $x^2 < \frac{1}{4}$ which implies that the radius of convergence is $1/2$.

10. [10 points] Determine whether the following series converge or diverge. Justify your answers.

a. [5 points] $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$

Solution: Note that this series is alternating and that the absolute values of the terms $\dfrac{|\ln(n)|}{|n|}$ form a decreasing sequence that converges to 0 as n approaches ∞. Therefore, this series converges by the alternating series test.

b. [5 points] $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3}+2}$

Solution: This can be done with a comparison or limit comparison test. For comparison:

$$\frac{n}{\sqrt{n^3}+2} > \frac{n}{3\sqrt{n^3}} = \left(\frac{1}{3}\right)\frac{1}{\sqrt{n}}$$

By the p-test with $p = 1/2 < 1$, we have that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. By the comparison test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3}+2}$ also diverges.

11. [9 points] Circle all true statements.

a. [3 points] The integral $\displaystyle\int_0^1 \frac{1}{\sin(x)}dx$

I. converges by the comparison test because $\dfrac{1}{\sin(x)} \leq C$ for some constant C for $0 < x \leq 1$ and $\displaystyle\int_0^1 C dx$ converges.

II. $\boxed{\text{diverges by the comparison test because } \dfrac{1}{\sin(x)} \geq \dfrac{1}{x} \text{ for } 0 < x \leq 1 \text{ and } \displaystyle\int_0^1 \frac{1}{x}dx \text{ diverges.}}$

III. diverges because $\displaystyle\lim_{x\to 0} \frac{1}{\sin(x)} \neq 0$.

IV. converges by the alternating series test because the values of $\sin(x)$ oscillate between -1 and 1.

b. [3 points] The series $\displaystyle\sum_{n=0}^{\infty} \frac{e^{n^2}}{n!}$

I. converges because $\displaystyle\lim_{n\to\infty} \frac{e^{n^2}}{n!} = 0$.

II. converges because factorials grow faster than exponential functions.

III. $\boxed{\text{diverges by the ratio test.}}$

IV. $\boxed{\text{diverges by the comparison test because } \dfrac{e^{n^2}}{n!} \geq e^n \text{ for } n = 0, 1, 2, 3, \ldots \text{ and } \displaystyle\sum_{n=0}^{\infty} e^n \text{ diverges.}}$

c. [3 points] The differential equation $\dfrac{dy}{dt} = t(y-2)(\ln(y))$ defined for $t > 0$ and $y > 0$ has

I. an unstable equilibrium solution at $t = 0$.

II. a stable equilibrium solution at $y = 2$.

III. $\boxed{\text{a stable equilibrium solution at } y = 1.}$

IV. $\boxed{\text{an unstable equilibrium solution at } y = 2.}$

12. [8 points] Franklin, your friendly new neighbor, is building a large chicken sanctuary. You decide to help Franklin build a special chicken coop with volume (in cubic km) given by the integral

$$\int_0^1 x\sqrt{1-\cos(x^2)}\, dx.$$

This integral is difficult to evaluate precisely, so you decide to use the methods you've learned this semester to help out Franklin. Your friend and president-elect, Kazilla, stops by to give you a hand. She suggests finding the 4th degree Taylor polynomial, $P_4(x)$, for the function $1-\cos(x^2)$ near $x=0$.

a. [4 points] Find $P_4(x)$.

Solution: We can use the Taylor series expansion for $\cos(x^2)$

$$\cos(y) = \sum_{n=0}^{\infty} \frac{(-1)^n\, y^{2n}}{(2n)!} = 1 - \frac{y^2}{2!} + \cdots + \frac{(-1)^n\, y^{2n}}{(2n)!} + \cdots$$

so

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n\, (x^2)^{2n}}{(2n)!} = 1 - \frac{(x^2)^2}{2!} + \cdots + \frac{(-1)^n\, (x^2)^{2n}}{(2n)!} + \cdots$$

Therefore

$$P_4 = 1 - (1 - \frac{x^4}{2!}) = \frac{x^4}{2!}$$

b. [4 points] Substitute $P_4(x)$ for $1-\cos(x^2)$ in the integral and compute the resulting integral by hand, showing all of your work.

Solution:

$$\begin{aligned}\int_0^1 x\sqrt{P_4(x)}dx &= \int_0^1 x\sqrt{\frac{x^4}{2}} \\ &= \int_0^1 \frac{x^3}{\sqrt{2}} \\ &= \frac{x^4}{4\sqrt{2}}\Big|_{x=0}^{1} \\ &= \frac{1}{4\sqrt{2}}\end{aligned}$$

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$