Math 116 — First Midterm

October 14, 2015

UMID: EXAM SOLUTIONS Initials: ____________

Instructor: ____________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	16	
2	7	
3	9	
4	10	
5	11	
6	7	
7	9	
8	13	
9	10	
10	8	
Total	100	

1. [16 points] Suppose $g(x)$ is a function with the following properties:

- $\displaystyle\int_5^1 g(x)dx = 7.$
- $\displaystyle\int_3^5 g(x)dx = -3.$
- $g(x)$ is odd.

In addition, a table of values for $g(x)$ is given below.

x	0	1	2	3	4	5
$g(x)$	0	2	-1	-3	-1	1

Calculate (a)-(c) **exactly**. Show your work and do not write any decimal approximations.

a. [4 points] $\displaystyle\int_1^{\sqrt{3}} xg(x^2)dx.$

Solution: Using the substitution $w = x^2, dw = 2xdx$, $\int_1^{\sqrt{3}} xg(x^2)dx = \int_1^3 \frac{1}{2}g(w)dw = \frac{1}{2}\left(\int_1^5 g(w)dw + \int_5^3 g(w)dw\right) = \frac{1}{2}(-7+3) = -2.$

b. [4 points] $\displaystyle\int_1^5 xg'(x)dx.$

Solution: Using integration by parts, $u = x, dv = g'(x)dx$, so $du = dx, v = g(x)$, the integral $\int_1^5 xg'(x)dx = [xg(x)]_1^5 - \int_1^5 g(x)dx = (5g(5) - g(1)) - (-7) = 10.$

c. [3 points] The average value of $g(x)$ on $[-5, -1]$.

Solution: The average value is $\frac{1}{-1-(-5)}\int_{-5}^{-1} g(x)dx$. Since g is odd, this is equal to $\frac{-1}{4}\int_1^5 g(x)dx = \frac{-1}{4}(-7) = \frac{7}{4}.$

d. [5 points] Approximate

$$\int_2^4 xg(x)dx$$

using TRAP(2). Write out all the terms of your sum and your final answer.

Solution:

$$\text{LEFT}(2) = \frac{4-2}{2}(2g(2) + 3g(3)) = -11$$
$$\text{RIGHT}(2) = \frac{4-2}{2}(3g(3) + 4g(4)) = -13$$
$$\text{TRAP}(2) = \frac{1}{2}(\text{LEFT}(2) + \text{RIGHT}(2)) = -12.$$

2. [7 points] Suppose

$$G(x) = \int_{2x^3}^{1/4} \cos^2(t^2)dt.$$

a. [3 points] Calculate $G'(x)$.

Solution: By the Fundamental Theorem of Calculus, $G'(x) = -6x\cos^2(4x^6)$.

b. [4 points] Find a constant a and a function h so that

$$G(x) = \int_a^x h(t)dt.$$

Solution: By the Fundamental Theorem of Calculus, $G'(x) = h(x)$, so $h(t) = -6t^2\cos^2(4t^6)$. Using the substitution $w = 2t^3, dw = 6t^2$,

$$G(x) = \int_a^x -6t^2\cos^2(4t^6)dt = -\int_{2a^3}^{2x^3} \cos^2(w^2)dw$$

so $a = \frac{1}{2}$.

$a =$ $\frac{1}{2}$

$h(t) =$ $-6t^2\cos^2(4t^6)$

3. [9 points] For each of the following questions, fill in the blank with the letter corresponding to the correct answer from the bottom of the page. No credit will be given for unclear answers.

a. [3 points] The length of the curve $y = \pi e^{-x}$ between $x = 2$ and $x = 4$ is:

H

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b. [3 points] Consider the region bounded by the x-axis, $y = -2e^{-2x}$, $x = 2$, and $x = 4$. The volume of the solid whose base is that region and whose cross sections perpendicular to the x-axis are semicircles is:

A

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c. [3 points] The volume of the solid obtained by rotating the region bounded by $y = e^{-x}$, $y = -2e^{-2x}$, $x = 2$ and $x = 4$, around the y-axis is:

C

(A) $\displaystyle\int_2^4 \frac{\pi}{2} e^{-4x} dx$

(B) $\displaystyle\int_2^4 \pi(e^{-2x} - 4e^{-4x}) dx$

(C) $\displaystyle\int_2^4 2\pi x(e^{-x} + 2e^{-2x}) dx$

(D) $\displaystyle\int_2^4 \pi\sqrt{1 + e^{-2x}} dx$

(E) $\displaystyle\int_2^4 2\pi e^{-4x} dx$

(F) $\displaystyle\int_2^4 2\pi\sqrt{1 + e^{-4x}} dx$

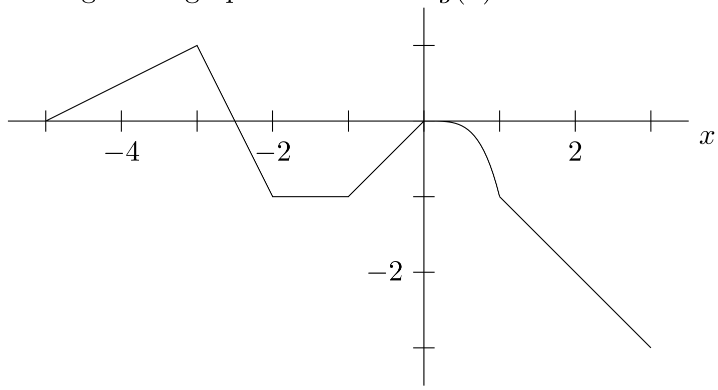
(G) $\displaystyle\int_2^4 \frac{\pi}{2} e^{-2x} dx$

(H) $\displaystyle\int_2^4 \sqrt{1 + \pi^2 e^{-2x}} dx$

(I) $\displaystyle\int_2^4 2\pi(e^{-2x} - 2e^{-4x}) dx$

(J) $\displaystyle\int_2^4 2\pi(e^{-x} + 2e^{-2x}) dx$

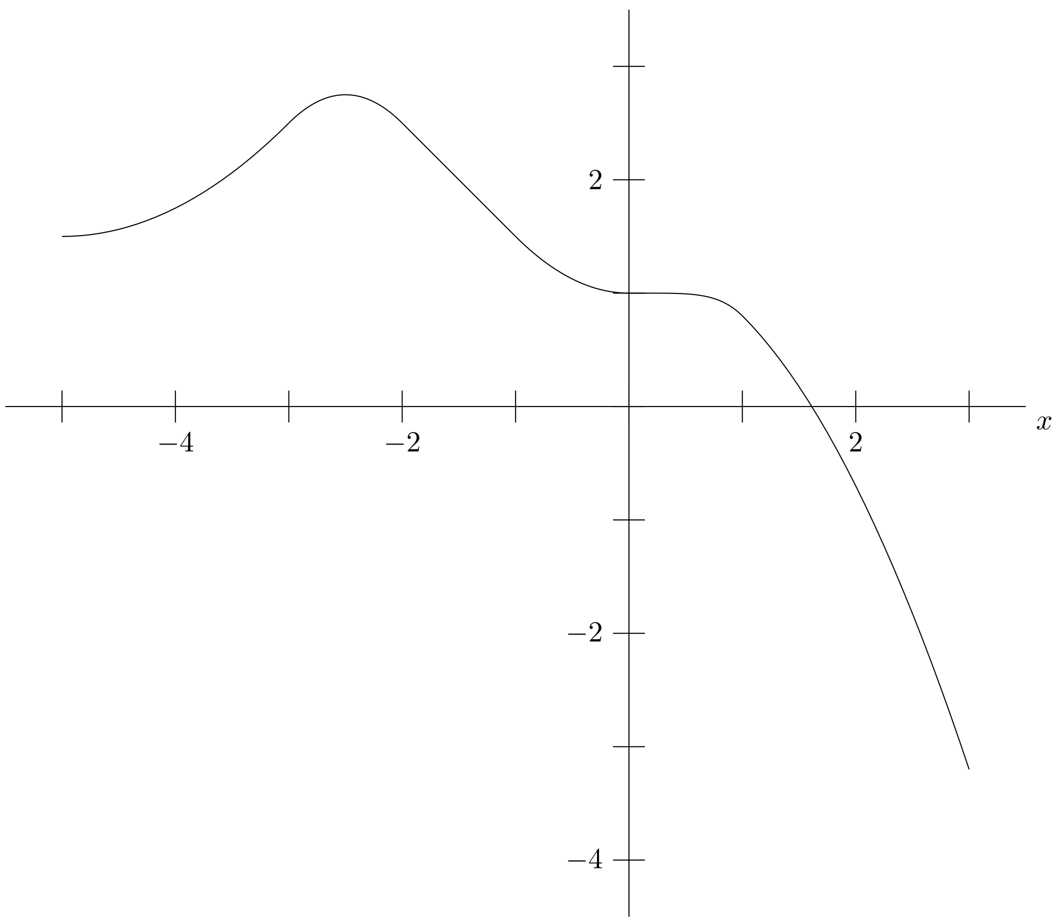
4. [10 points] The following is the graph of a function $g(x)$.



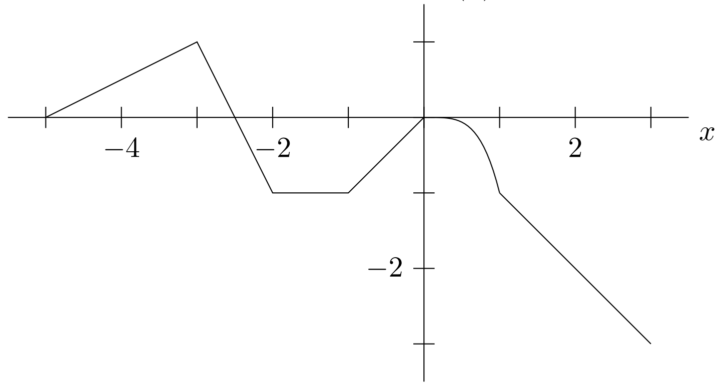
Note that $g(x)$ is piecewise linear on [-5,0] and linear on [1,3].

Let $G(x)$ be an antiderivative of $g(x)$ such that $G(0) = 1$. Carefully sketch a graph of $G(x)$ using the axes provided below. In the blanks provided, give the values of $G(-5)$ and $G(-1)$ and be sure your graph is consistent with these values.

$G(-5) =$ $\underline{\qquad \frac{3}{2} \qquad}$ $G(-1) =$ $\underline{\qquad \frac{3}{2} \qquad}$



5. [11 points] The following is the graph of a function $g(x)$.



Note that $g(x)$ is piecewise linear on [-5,0] and linear on [1,3].

a. [5 points] Estimate $\displaystyle\int_1^3 e^{-g(x)}dx$ using MID(2). Write out all the terms of your sum as well as your final answer.

Solution:

$$\text{MID}(2) = \frac{3-1}{2}\left(e^{-g(1.5)} + e^{-g(2.5)}\right) = e^{1.5} + e^{2.5} \approx 16.664$$

b. [3 points] If you were estimating $\displaystyle\int_0^1 \frac{dx}{g(x)+2}$ using LEFT(n), would your estimate be an overestimate, underestimate, or is there not enough information to tell? Circle your answer. You do not need to show your work or explain your answer.

Overestimate $\boxed{\text{Underestimate}}$ Not Enough Information

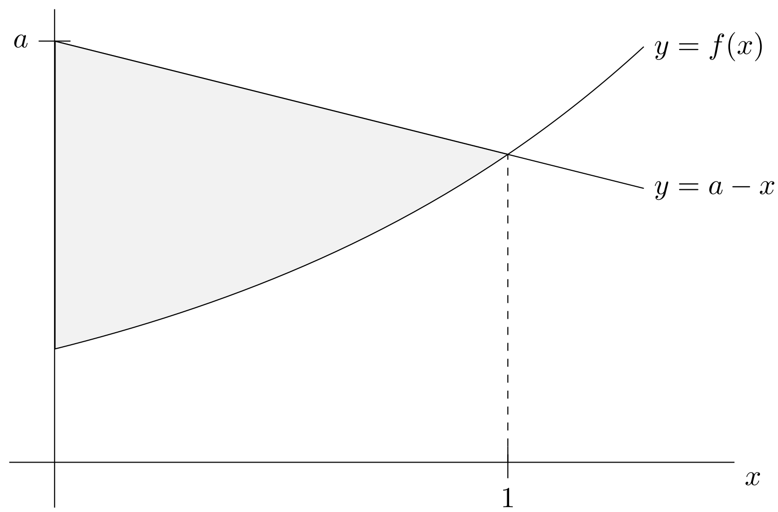
Solution: $\dfrac{d}{dx}\left(\dfrac{1}{g(x)+2}\right) = \dfrac{-g'(x)}{(g(x)+2)^2} > 0$, so LEFT($n$) is an underestimate.

c. [3 points] If you were estimating $\displaystyle\int_0^1 \frac{dx}{g(x)+2}$ using TRAP(n), would your estimate be an overestimate, underestimate, or is there not enough information to tell? Circle your answer. You do not need to show your work or explain your answer.

$\boxed{\text{Overestimate}}$ Underestimate Not Enough Information

Solution: $\dfrac{d^2}{dx^2}\left(\dfrac{1}{g(x)+2}\right) = \dfrac{-g''(x)(g(x)+2)^2 + 2(g'(x))^2(g(x)+2)}{(g(x)+2)^4} > 0$, so TRAP($n$) is an overestimate.

6. [7 points] A thin plate has the shape bounded by the curves $y = f(x)$, $y = a - x$ and the y-axis, for some positive constant a. The plate has a density of $\delta(x) = 7 + x$ kg/m^2.



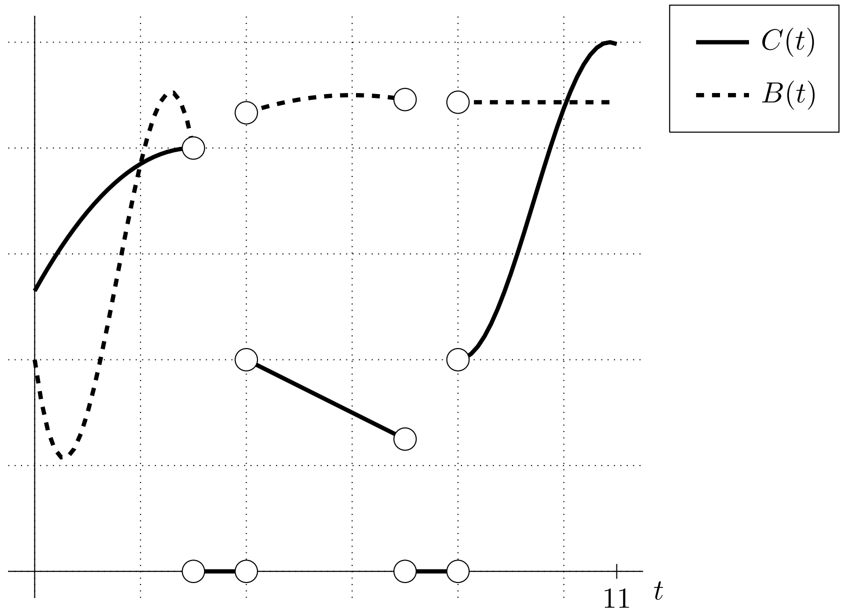
a. [4 points] Write an expression involving integrals that represents the total mass of the plate. Do not evaluate any integrals.

Solution: A thin vertical slice x m from the y-axis has a mass of $((a-x)-f(x))\delta(x)\Delta x$, so the total mass is $\displaystyle\int_0^1 ((a-x)-f(x))(7+x)dx$ kg.

b. [3 points] Write an expression involving integrals that represents the x-coordinate of the center of mass of the plate. Do not evaluate any integrals.

Solution: $\overline{x} = \dfrac{\int_0^1 x((a-x)-f(x))(7+x)dx}{\int_0^1((a-x)-f(x))(7+x)dx}$ meters.

7. [9 points] Boxers Paul "Stretch" Cassenick and Stephen "Dee" Boxer decide to rematch for the heavyweight title. Suppose that the match consists of three 3-minute rounds with a 1-minute break in between each round. Suppose $C(t)$ is a function which models the number of punches Paul throws per minute, t minutes after the start of the match. Similarly, suppose $B(t)$ models the number of punches Stephen throws per minute, t minutes after the start of the match. Assume all punches thrown in the match are taken by the opponent.



a. [3 points] Find an expression involving integrals that represents the average number of punches thrown per minute by Stephen t minutes after the fight has started.

Solution: The average value of B on the interval $[0, t]$ is $\frac{1}{t}\int_0^t B(p)dp$.

b. [2 points] Paul's scoring margin at time t is defined to be

(Total punches thrown by Paul at time t) − (total punches taken by Paul at time t).

At approximately what time(s) is Paul's scoring margin the largest? The smallest?

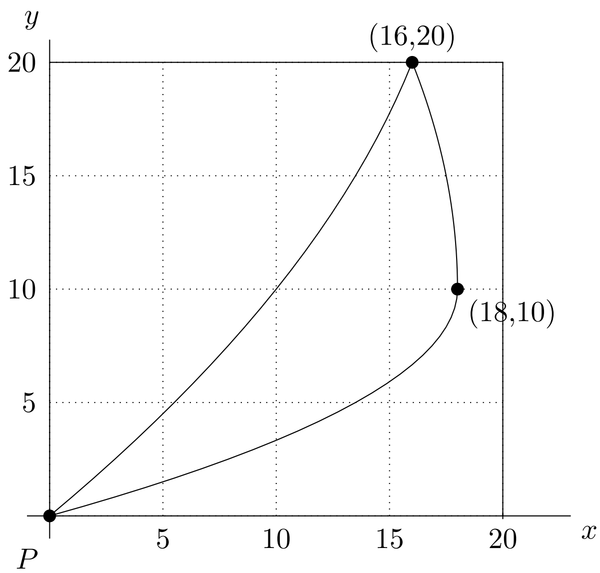
Largest at $t =$ **2**

Smallest at $t =$ **10**

c. [4 points] Assuming that the winner is the boxer who threw the most punches, who wins the fight? Give a brief justification of your answer making reference to the graph.

Solution: Based on the graphs, $\int_0^{11} B(t)dt > \int_0^{11} C(t)dt$, so Stephen is the winner of the fight.

8. [13 points] During the first round of the rematch between Paul "Stretch" Cassenick and Stephen "Dee" Boxer, Paul's position in the boxing ring t minutes after the 3-minute round began is given by $(x(t), y(t))$ where $x(t)$ and $y(t)$ are Paul's distance from his corner, in feet, in the $x-$ and $y-$directions, respectively. The ring is the 20x20 foot square pictured below, and the point P is Paul's corner. Suppose $x(t) = -8t(t-3)$, and $y(t)$ has values given in the table below and is **linear** between each consecutive pair of t-values in the table.



t	$y(t)$
0	0
1	20
1.5	10
3	0

a. [5 points] On the diagram of the ring, sketch a graph of Paul's path through the ring during the first round of the rematch. Label the points corresponding to Paul's position at $t = 1$ and $t = 1.5$ with their x- and y-coordinates.

b. [4 points] Find the slope of the tangent line to Paul's path at $t = 2$.

Solution: $\dfrac{dy}{dx} = \dfrac{\left(\frac{dy}{dt}\big|_{t=2}\right)}{\left(\frac{dx}{dt}\big|_{t=2}\right)} = \dfrac{\frac{-20}{3}}{-8} = \dfrac{5}{6}.$

c. [4 points] Write an explicit expression involving integrals that gives the distance Paul traveled during the first minute of the round. Your answer should not contain the letters 'x' or 'y'.

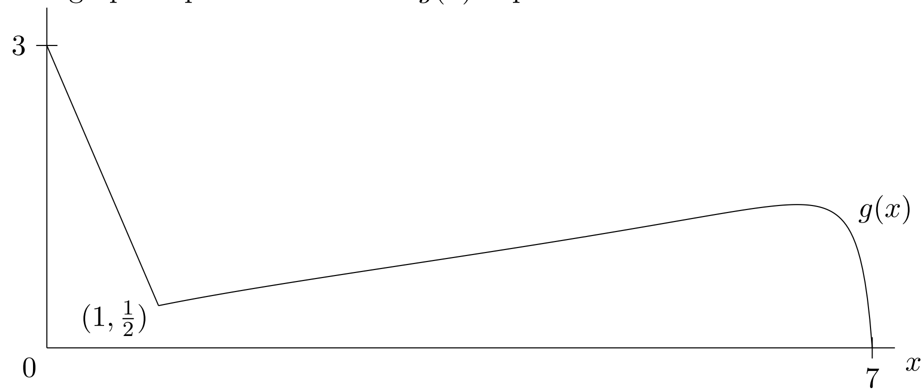
Solution: The distance traveled over the first minute is the arc length of the curve from $t = 0$ to $t = 1$. So the distance Paul traveled is $\int_0^1 \sqrt{(24-16t)^2 + (20)^2}dt$ feet.

9. [10 points] Tracy Johnson is taking the trash out of her apartment on State Street. In order to throw the trash bag into the dumpster, she lifts the bag at a constant speed from the ground up to a height of 2 meters. The bag of trash weighs 5 kg initially, but an unfortunate hole causes the bag to leak trash at a constant rate. Recall that the gravitational constant is $g = 9.8$ m/s^2.

If the bag weighs 3 kg when it is one meter from the ground, compute the work required for Tracy to lift the bag of trash into the dumpster. Evaluate **by hand** any integrals you compute.

Solution: The trash bag is leaking at a constant rate, so the weight of the bag is a linear function of the height above the ground. Using the information, the bag has a mass of $5 - 2h$ kg at a height of h meters. Since the force on the bag at height h is $(5 - 2h)g$, the total work is $\int_0^2 (5 - 2h)g\, dh = 6g = 58.8$ joules.

10. [8 points] The graph of part of a function $g(x)$ is pictured below.



a. [4 points] A thumbtack has the shape of the solid obtained by rotating the region bounded by $y = g(x)$, the x-axis and y-axis, about the y-axis. Find an expression involving integrals that gives the volume of the thumbtack. Do not evaluate any integrals.

Solution: Using the cylindrical shell method, the volume of the thumbtack is $\int_0^7 2\pi x g(x)$.

b. [4 points] A door knob has the shape of the solid obtained by rotating the region bounded by $y = g(x)$, the x-axis and y-axis, about the x-axis. Find an expression involving integrals that gives the volume of the door knob. Do not evaluate any integrals.

Solution: Using the washer method, the volume of the door knob is $\int_0^7 \pi(g(x))^2 dx$.