Math 116 — Second Midterm

November 18, 2015

UMID: EXAM SOLUTIONS Initials: ____________

Instructor: ____________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 9 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	12	
2	8	
3	6	
4	10	
5	10	
6	15	
7	18	
8	9	
9	12	
Total	100	

1. [12 points] Determine if the following integrals converge or diverge. If the integral converges, circle the word "converges" and give the **exact value** (i.e. no decimal approximations). If the integral diverges, circle "diverges". In either case, **you must show all your work and indicate any theorems you use**. Any direct evaluation of integrals must be done **without using a calculator**.

a. [6 points] $\displaystyle\int_0^1 \frac{e^x \sin(2x) - (2e^x - 2)\cos(2x)}{\sin^2(2x)}\,dx$

$$\left(\text{Note: } \frac{d}{dx}\left(\frac{e^x - 1}{\sin(2x)}\right) = \frac{e^x \sin(2x) - (2e^x - 2)\cos(2x)}{\sin^2(2x)}\right)$$

Converges Diverges

Solution:

$$\begin{aligned}\int_0^1 \frac{e^x \sin(2x) - (2e^x - 2)\cos(2x)}{\sin^2(2x)}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{e^x \sin(2x) - (2e^x - 2)\cos(2x)}{\sin^2(2x)}\,dx\\ &= \lim_{a\to 0^+}\left[\frac{e^x - 1}{\sin(2x)}\right]_a^1\\ &= \lim_{a\to 0^+}\left(\frac{e-1}{\sin(2)} - \frac{e^a - 1}{\sin(2a)}\right)\\ &= \frac{e-1}{\sin(2)} - \lim_{a\to 0^+}\left(\frac{e^a}{2\cos(2a)}\right) \quad \text{by L'Hopital's Rule}\\ &= \frac{e-1}{\sin(2)} - \frac{1}{2}.\end{aligned}$$

b. [6 points] $\displaystyle\int_2^\infty \frac{1}{(\ln x)^2 x}\,dx$

Converges Diverges

Solution:

$$\int_2^\infty \frac{1}{(\ln x)^2 x}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{(\ln x)^2 x}\,dx$$

Using the substitution $w = \ln x, dw = \frac{1}{x}dx$, the integral becomes

$$\begin{aligned}\lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{1}{w^2}\,dw &= \lim_{b\to\infty}\left[\frac{-1}{w}\right]_{\ln 2}^{\ln b}\\ &= \lim_{b\to\infty}\left(\frac{-1}{\ln b} - \frac{-1}{\ln 2}\right)\\ &= \frac{1}{\ln 2}\end{aligned}$$

2. [8 points] Wild rabbits were introduced to Australia in 1859. The behavior of the rabbit population P in Australia at a time t years after 1859 was modeled by the differential equation

$$P' = P + e^{-t}.$$

a. [4 points] For what value of B is

$$P = 3e^t + Be^{-t}$$

a solution to the differential equation? Be sure to show clearly how you got your answer.

Solution: We can compute that

$$P' = 3e^t - Be^{-t},$$

so if $P = 3e^t - Be^{-t}$ is a solution to the differential equation,

$$3e^t - Be^{-t} = 3e^t + Be^{-t} + e^{-t}.$$

Solving, we see that $B = \frac{-1}{2}$.

b. [4 points] Suppose that the rabbit population in 1859 was 24 rabbits. Historians used Euler's method with $\Delta t = \frac{1}{2}$ to estimate the rabbit population in 1861. Is their answer an overestimate or and underestimate? Give a brief justification of your answer.

Overestimate **Underestimate**

Solution: Taking a derivative of the differential equation, we see that

$$P'' = P' - e^{-t} = (P + e^{-t}) - e^{-t} = P..$$

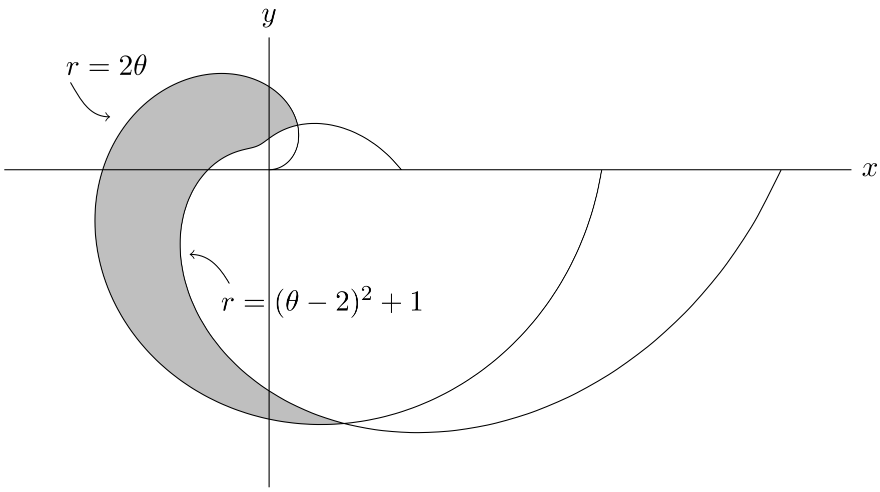
Since $P(0) = 24 > 0$ and $P' = P + e^{-t} > 0$, P is always positive. Thus $P'' = P > 0$ and so P is concave up, hence Euler's method gives an underestimate.

3. [6 points] Write an explicit expression involving integrals which gives the arc length of **one petal** of the polar rose $r = 3\cos(5\theta)$. Your answer should not contain the letter 'r'. Do not evaluate any integrals.

Solution: The arc length of one petal is

$$\int_{-\pi/10}^{\pi/10} \sqrt{(3\cos(5\theta))^2 + (-15\sin(5\theta))^2}d\theta.$$

4. [10 points] The visible portion of the strangely-shaped moon of the planet Thethis during its waxing crescent phase is in the shape of the region bounded between the polar curves $r = 2\theta$ and $r = (\theta-2)^2+1$. The region is pictured below. Assume x and y are measured in thousands of miles.



a. [6 points] Write an expression involving integrals which gives the area of the visible portion of this moon. Include the units of the integral in your answer. Do not evaluate any integrals.

Solution: The two curves intersect when $\theta = 1, 5$. Therefore the area of the moon is

$$\left(\int_1^5 \frac{1}{2}(2\theta)^2 d\theta - \int_1^5 \frac{1}{2}((\theta-2)^2+1)^2 d\theta\right) \text{ (thousand miles)}^2$$
$$= \left(\int_1^5 \frac{1}{2}(2\theta)^2 d\theta - \int_1^5 \frac{1}{2}((\theta-2)^2+1)^2 d\theta\right) \text{ million miles}^2.$$

b. [4 points] Find the slope of the tangent line to the polar curve $r = (\theta-2)^2+1$ at $\theta = \pi$.

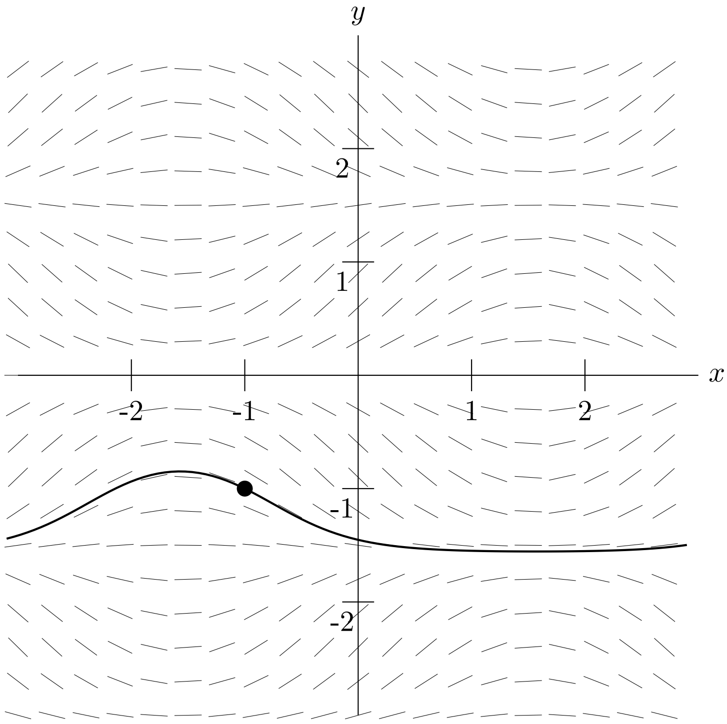
Solution: Converting to parametric equations, we have

$$x = r\cos\theta = ((\theta-2)^2+1)\cos\theta$$
$$y = r\sin\theta = ((\theta-2)^2+1)\sin\theta$$

Thus

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{\left.\frac{dy}{d\theta}\right|_{\theta=\pi}}{\left.\frac{dx}{d\theta}\right|_{\theta=\pi}} = \frac{2(\pi-2)\sin\pi + ((\pi-2)^2+1)\cos\pi}{2(\pi-2)\cos\pi - ((\pi-2)^2+1)\sin\pi} = \frac{(\pi-2)^2+1}{2(\pi-2)}.$$

5. [10 points] The graph of a slope field corresponding to a differential equation is shown below.



a. [4 points] On the slope field, carefully sketch a solution curve passing through the point (-1,-1).

Solution: See graph above.

b. [2 points] The slope field pictured above is the slope field for one of the following differential equations. Which one? Circle your answer. You do not need to show your work.

$$\frac{dy}{dx} = \cos x \cos(2y) \qquad \frac{dy}{dx} = \sin x \cos(2y)$$

$$\boxed{\frac{dy}{dx} = \cos x \sin(2y)} \qquad \frac{dy}{dx} = \sin x \sin(2y)$$

c. [4 points] Find two equilibrium solutions to the differential equation you circled.

Solution: The equilibrium solutions of $\frac{dy}{dx} = \cos x \sin(2y)$ are the values of y such that $\sin(2y) = 0$. Solving, we see that the equilibrium solutions are $y = 0, \pm\frac{\pi}{2}, \pm\pi, \ldots$

6. [15 points] For each of the following questions, fill in the blank with the letter corresponding to the answer from the bottom of the page that correctly completes the sentence. No credit will be given for unclear answers. You do not need to show your work.

a. [3 points] The limit, $\lim_{x\to\infty}\left(\frac{x+2}{x}\right)^{x/2}$,...

B

b. [3 points] The value of the integral $\int_{-1}^{1} \frac{e}{x^{1/3}}dx$...

G

c. [3 points] The value of the integral $\int_{-1}^{2} \frac{8e}{x^3}dx$...

J or I

d. [3 points] The value of A for which the differential equation $y'' = Ay$ is satisfied by the function $f(t) = e^{et}$...

C

e. [3 points] The length of the polar curve $r = \frac{4e}{\pi}\cos(\theta)$ between $\theta = -\pi/4$ and $\theta = \pi/4$...

D

(A)...is $e^{1/2}$.	(F)...is 1.
(B)...is e.	(G)...is 0.
(C)...is e^2.	(H)...is 2.
(D) ...is $2e$.	(I)...does not exist.
(E)...is $3e$.	(J)...diverges.

7. [18 points] A certain small country called Merrimead has 25 million dollars in paper currency in circulation, and each day 50 thousand dollars comes into Merrimead's banks. The government decides to introduce new currency by having the banks replace the old bills with new ones whenever old currency comes into the banks. Assume that the new bills are equally distributed throughout all paper currency. Let $M = M(t)$ denote the amount of new currency, in thousands of dollars, in circulation at time t days after starting to replace the paper currency.

a. [5 points] Write a differential equation involving $M(t)$, including an appropriate initial condition.

Solution: The concentration of old bills among all bills in circulation is $\frac{25000-M}{25000}$, and 50 thousand dollars moves through the bank each day, so

$$\frac{dM}{dt} = \text{Concentration} \times \text{Money per day} = \frac{25000 - M}{25000} \cdot 50, \qquad M(0) = 0.$$

Now consider the differential equation

$$B^2 + 2B\frac{dB}{dt} = 2500.$$

b. [4 points] Find all equilibrium solutions and classify their stability.

Solution: If $\frac{dB}{dt} = 0$ then we see that $B^2 = 2500$, so the equilibrium solutions are $B = \pm 50$. Both equilibrium solutions are stable.

Brightcrest, a second small country, also wants to replace all of their old paper bills as well, using a different strategy than Merrimead. The amount $B(t)$, in millions of dollars, of new paper currency in circulation in Brightcrest at a time t years after starting to replace the paper currency is modeled by the differential equation for B above with initial condition $B(0) = 0$.

c. [6 points] Find a formula for $B(t)$.

Solution: Using separation of variables, we have $\int \frac{2B}{2500-B^2}\, dB = \int dt$. Using the substitution $w = 2500 - B^2, dw = -2B\, dB$, we see that $-\ln|2500 - B^2| = t + C$. Solving for B, we see $B = \sqrt{2500 - Ae^{-t}}$.
Since $B(0) = 0$, we see that $A = 2500$, so

$$B(t) = \sqrt{2500 - 2500e^{-t}}.$$

d. [3 points] Assuming that all of the old bills are replaced in the long run, how much time will pass after starting to replace the paper bills until the new currency accounts for 99% of all currency in Brightcrest?

Solution: Since all of the bills are replaced in the long run, the total amount of money in circulation is $\lim_{t\to\infty} B(t) = 50$ million dollars. So the amount of time that passes until the new bills account for 99% of all currency is the value of t so that

$$(.99)(50) = \sqrt{2500 - 2500e^{-t}}.$$

So $t = -\ln(1 - (.99)^2) \approx 3.92$ years.

8. [9 points] Boxer Paul "Stretch" Cassenick decides to do some footwork training by practicing moving around the boxing ring. The ring is in the shape of a 20x20 foot square, and Paul's movement is modeled by the differential equation

$$\frac{dy}{dx} = f(x,y),$$

for some function $f(x,y)$. The following table gives some values of $f(x,y)$.

y \ x	0	5	10	15	20
0	3	1	1	0	-2
5	0	0	0	0	0
10	2	1	2	0	2
15	0	-1	-3	0	1
20	-2	0	-2	0	-3

a. [6 points] If $y(0) = 0$, use Euler's method with $\Delta x = 5$ to estimate $y(20)$. You must show your calculation for each step of Euler's method to receive full credit.

Solution: Given that $y(0) = 0$, using Euler's method with $\Delta x = 5$ gives

$$\begin{aligned} y(5) &\approx 0 + (5)(3) = 15 \\ y(10) &\approx 15 + (5)(-1) = 10 \\ y(15) &\approx 10 + (5)(2) = 20 \\ y(20) &\approx 20 + (5)(0) = 20 \end{aligned}$$

Thus $y(20) \approx 20$.

b. [3 points] Circle all of the following that **could** be equilibrium solutions.

$x = 5$ $\qquad$ $x = 15$ $\qquad$ $\boxed{y = 5}$ $\qquad$ $y = 15$

9. [12 points] Determine if the following integrals converge or diverge. If the integral converges, circle the word "converges". If the integral diverges, circle "diverges". In either case, **you must show all your work and indicate any theorems you use**. You do not need to calculate the value of the integral if it converges.

a. [6 points] $\displaystyle\int_1^\infty \frac{2+\sin x}{\sqrt{x+1}}\,dx$

Converges $\boxed{\textbf{Diverges}}$

Solution: $\dfrac{2+\sin x}{\sqrt{x+1}} \ge \dfrac{1}{\sqrt{2x}}$, so $\displaystyle\int_1^\infty \frac{2+\sin x}{\sqrt{x+1}}\,dx \ge \int_1^\infty \frac{1}{\sqrt{2x}}\,dx$.
But $\displaystyle\int_1^\infty \frac{1}{\sqrt{2x}}\,dx$ diverges by the p-test, $p = \frac{1}{2} \le 1$, so $\displaystyle\int_1^\infty \frac{2+\sin x}{\sqrt{x+1}}\,dx$ also diverges by the direct comparison test.

b. [6 points] $\displaystyle\int_1^\infty \frac{\theta}{\sqrt{\theta^5+1}}\,d\theta$

$\boxed{\textbf{Converges}}$ Diverges

Solution: $\dfrac{\theta}{\sqrt{\theta^5+1}} \le \dfrac{\theta}{\sqrt{\theta^5}} = \dfrac{1}{\theta^{3/2}}$, so $\displaystyle\int_1^\infty \frac{\theta}{\sqrt{\theta^5+1}}\,d\theta \le \int_1^\infty \frac{1}{\theta^{3/2}}\,d\theta$.
But $\displaystyle\int_1^\infty \frac{1}{\theta^{3/2}}\,d\theta$ converges by the p-test, $p = \frac{3}{2} > 1$, so $\displaystyle\int_1^\infty \frac{\theta}{\sqrt{\theta^5+1}}\,d\theta$ also converges by the direct comparison test.