Math 116 — Final Exam

December 17, 2015

UMID: EXAM SOLUTIONS Initials: ____________

Instructor: ____________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 13 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	10	
2	8	
3	13	
4	8	
5	3	
6	4	
7	6	
8	9	
9	4	
10	4	
11	7	
12	9	
13	10	
Total	95	

1. [10 points] Paul "Stretch" Cassenick and Stephen "Dee" Boxer never did have a third boxing match. Stephen chose a life of pacifism at a monastery in a far off country, and Paul became the czar of a Calculus 2 course at a large public university.

a. [6 points] To stay awake on the day of the final exam for his Calculus 2 class, Paul uses an intravenous drip of caffeine. Some values for the rate at which Paul is being given caffeine, $r(t)$, in mg/hour, t hours after he wakes up on the day of the exam are given in the table below. Assume the function $r(t)$ is differentiable and that its concavity does not change.

t	0	2	4	6	8
$r(t)$	95	68	50	40	35

Approximate the amount of caffeine given to Paul during the first 8 hours after he wakes up on the day of the exam using MID(n) with as many subdivisions as possible given the data in the table. Write out all the terms in your sum. Indicate whether your approximation is an over- or underestimate by circling the correct answer.

overestimate **underestimate**

Solution: The maximum number of subdivisions for MID is 2, and

$$\begin{aligned}\text{MID}(2) &= (4)(r(2) + r(6)) \\ &= (4)(68 + 40) \\ &= 432 \text{ mg}\end{aligned}$$

Since $r'(t)$ is increasing, $r(t)$ must be concave up. Thus MID(2) gives an underestimate.

b. [4 points] As part of his way to enlightenment, Stephen is digging a hole in the ground which will be the foundation for the new temple at his monastery. Suppose the soil has density $\delta(x)$ kg/m^3 at a depth of x meters below ground level at the temple site. If Stephen is digging a circular hole with diameter 60 meters, write an expression involving integrals which represents the mass of soil he has dug through when the hole has depth h meters.

Solution: A thin slice of the dirt in the hole of thickness δx meters has a mass of $\pi(30)^2\delta(x)$ kg. So the total mass of soil removed from the hole at a depth of h meters is

$$\int_0^h \pi(30)^2\delta(x)dx \text{ kg.}$$

2. [8 points] Let $f(x) = x^{2x}$. The first two derivatives of f are given below.

$$f'(x) = 2(1 + \ln x)x^{2x}$$
$$f''(x) = 2x^{2x-1} + 4(1 + \ln x)^2 x^{2x}$$

a. [4 points] Find the 2nd degree Taylor polynomial $P_2(x)$ of f centered at $x = 1$.

Solution: Using the formula for Taylor polynomials,

$$\begin{aligned} P_2(x) &= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= 1 + 2(x-1) + 3(x-1)^2 \end{aligned}$$

$P_2(x) =$ $1 + 2(x-1) + 3(x-1)^2$

b. [4 points] Find

$$\lim_{x\to 1} \frac{x^{2x} - 1}{3x - 3}.$$

Clearly show your reasoning. Your answer from part **(a)** may be helpful.

Solution:

$$\begin{aligned} \lim_{x\to 1} \frac{x^{2x} - 1}{3x - 3} &= \lim_{x\to 1} \frac{1 + 2(x-1) + 3(x-1)^2 - 1}{3(x-1)} \\ &= \lim_{x\to 1} \frac{2 + 3(x-1)}{3} \\ &= \frac{2}{3} \end{aligned}$$

3. [13 points]

a. [4 points] The number of people R that have heard a rumor increases at a rate proportional to the product of the number of people that have heard the rumor and the number of people that haven't yet heard the rumor. Write a differential equation for R which models the scenario described assuming that the total number of people is 1,000. Use $k > 0$ for the constant of proportionality.

Solution: The number of people that have heard the rumor is R, so the number of people that have not yet heard the rumor is $1000 - R$. Thus the differential equation is $\frac{dR}{dt} = kR(1000 - R)$.

$$\frac{dR}{dt} = \underline{\qquad kR(1000 - R) \qquad}$$

b. [4 points] For what values of A, B is $y(t) = At\cos t + Bt$ a solution to the differential equation $ty' = y + t^2 \sin t$ satisfying the initial condition $y\left(\frac{\pi}{2}\right) = 2\pi$? Be sure to show your work.

Solution: Since $y'(t) = A\cos t - At\sin t + B$, $y(t)$ is a solution if

$$t(A\cos t - At\sin t + B) = At\cos t + Bt + t^2 \sin t \qquad \Rightarrow \qquad -At^2 \sin t = t^2 \sin t$$

Thus $A = -1$. Plugging in the initial condition $y\left(\frac{\pi}{2}\right) = 2\pi$,

$$-\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + B\left(\frac{\pi}{2}\right) = 2\pi.$$

Thus $B = 4$.

$A = $ **-1**

$B = $ **4**

c. [5 points] Find the solution to the differential equation

$$e^{-x} + y^2\frac{dy}{dx} = 0, \quad \text{with initial condition } \ y(0) = 2.$$

Solution: Moving the e^{-x} to the right side of the equation and separating variables,

$$\int y^2 dy = \int -e^{-x} dx$$
$$\frac{1}{3}y^3 = e^{-x} + C$$
$$y = \sqrt[3]{3e^{-x} + C}$$

Plugging in the initial condition $y(0) = 2$, $2 = \sqrt[3]{3 + C}$. Therefore $C = 5$.

$$y = \underline{\qquad \sqrt[3]{3e^{-x} + 5} \qquad}$$

4. [8 points] Let $f(x) = \sqrt[3]{1+2x^2}$.

a. [5 points] Find the first 3 nonzero terms of the Taylor series for f centered at $x = 0$.

Solution: Using the Taylor series for $(1+y)^{1/3}$ centered at $y = 0$,

$$\begin{aligned}\sqrt[3]{1+y} &= 1 + \frac{1}{3}y + \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)}{2!}y^2 + \dots \\ &= 1 + \frac{y}{3} - \frac{y^2}{9} + \dots\end{aligned}$$

Substituting $y = 2x^2$,

$$\sqrt[3]{1+2x^2} = 1 + \frac{2x^2}{3} - \frac{4x^4}{9} + \dots$$

b. [3 points] For what values of x does the Taylor series converge?

Solution: The binomial series for $\sqrt[3]{1+y}$ converges when $-1 < y < 1$. Substituting $y = 2x^2$, this converges when $1 < 2x^2 < 1$, or $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

5. [3 points] Determine the **exact** value of the infinite series

$$-1 + \frac{1}{3!} - \frac{1}{5!} + \dots + \frac{(-1)^{n+1}}{(2n+1)!} + \dots$$

No decimal approximations are allowed. You **do not** need to show your work.

Solution:

$$-1 + \frac{1}{3!} - \frac{1}{5!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{(2n+1)!} = \sin(-1).$$

6. [4 points] For each of the following questions, circle the answer which correctly completes the statement. You **do not** need to show your work.

a. [2 points] The integral $\displaystyle\int_1^\infty \frac{\ln x}{x^{3/2}}dx$

converges **diverges**

Solution: $\ln x \le x^{1/4}$ for sufficiently large values of x, so

$$\frac{\ln x}{x^{3/2}} \le \frac{x^{1/4}}{x^{3/2}} = \frac{1}{x^{5/4}}$$

eventually. Since $\int_1^\infty \frac{1}{x^{5/4}}\,dx$ converges by the p-test ($p = \frac{5}{4} > 1$), $\int_1^\infty \frac{\ln x}{x^{3/2}}\,dx$ also converges by direct comparison.

b. [2 points] The integral $\displaystyle\int_0^1 \frac{x}{x^2 + x^{3/2}}dx$

converges **diverges**

Solution:

$$\frac{x}{x^2 + x^{3/2}} = \frac{1}{x + \sqrt{x}} \le \frac{1}{\sqrt{x}}$$

for all positive values of x. Since $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges by the p-test ($p = \frac{1}{2} < 1$), $\int_0^1 \frac{x}{x^2+x^{3/2}}\,dx$ also converges by direct comparison.

7. [6 points] The power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{5^n n}$$

has a radius of convergence of 5. For each of the endpoints of the interval of convergence, fill in the first two blanks with the endpoint and the series at that endpoint (in sigma notation or by writing out the first 4 terms), and then indicate whether the series converges at that endpoint in the final blank. You **do not** need to show your work.

At the endpoint $x =$ **-4**, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$

and that series **diverges**.

At the endpoint $x =$ **6**, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

and that series **converges**.

8. [9 points] Consider the function $g(x)$ defined by the power series

$$g(x) = \sum_{n=0}^{\infty} \frac{2^n (n!)^2 x^n}{(2n)!}.$$

a. [6 points] Find the **radius** of convergence of the power series. You do not need to find the interval of convergence.

Solution: Applying the ratio test

$$\lim_{n\to\infty} \left| \frac{\left(\frac{2^{n+1}((n+1)!)^2 x^{n+1}}{(2(n+1))!}\right)}{\left(\frac{2^n (n!)^2 x^n}{(2n)!}\right)} \right| = \lim_{n\to\infty} \frac{2(n+1)^2|x|}{(2n+2)(2n+1)}$$
$$= \frac{|x|}{2}.$$

Therefore the series converges whenever $\frac{|x|}{2} < 1$, or $|x| < 2$. Hence the radius of convergence is 2.

b. [3 points] Use the first 3 nonzero terms of the power series to estimate

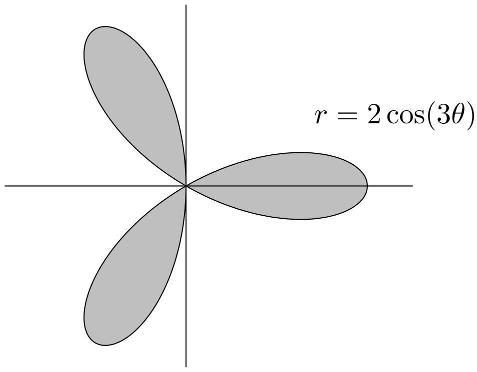
$$\int_0^1 \frac{g(x) - 1}{x}\, dx.$$

Solution: Since $g(x) = 1 + \frac{2}{2!}x + \frac{2^2(2!)^2}{4!}x^2 + \cdots = 1 + x + \frac{2}{3}x^2 + \ldots,$

$$\int_0^1 \frac{g(x)-1}{x}\, dx \approx \int_0^1 \frac{1 + x + \frac{2}{3}x^2 - 1}{x}\, dx = \int_0^1 (1 + \frac{2}{3}x)\, dx = \frac{4}{3}.$$

9. [4 points] Maria wants to get a tattoo of her favorite geometric shape. The shape happens to be the region enclosed by the rose $r = 2\cos(3\theta)$. In order to make sure that the tattoo turns out perfectly, the artist needs to know how much ink is necessary. Find an expression involving integrals that gives the total area of the shaded region depicted below.

Solution: Since $r = 2\cos(3\theta)$ completes one full cycle in π radians, the total area is $\frac{1}{2}\int_0^{\pi}(2\cos(3\theta))^2\,d\theta$.



10. [4 points] The series

$$\sum_{n=0}^{\infty} \frac{9^n}{8^n + 10^n}$$

converges.

Use an appropriate series test to show that the series converges. Be sure to indicate which test(s) you are using. Also verify all hypotheses needed for the test, and justify the convergence/divergence of any other series you use.

Solution: $\frac{9^n}{8^n+10^n} \le \frac{9^n}{10^n} = \left(\frac{9}{10}\right)^n$ for all values of n. Since $\displaystyle\sum_{n=0}^{\infty}\left(\frac{9}{10}\right)^n$ is a geometric series with $|r| = \frac{9}{10} < 1$, it converges. Thus $\displaystyle\sum_{n=0}^{\infty}\frac{9^n}{8^n+10^n}$ also converges by the Direct Comparison Test.

11. [7 points] Two squirrels, Zini and Aladar, are quickly scavenging for their last acorns before returning to their dens for winter. At a time t seconds after they begin running, Zini's position on the diag is given by

$$x(t) = t, \qquad y(t) = t - 3$$

and Aladar's position is given by

$$x(t) = 4t, \qquad y(t) = t^2$$

for $0 \le t \le 5$. Assume $x(t)$ and $y(t)$ are measured in meters.

a. [3 points] Find Aladar's **speed** 1 second after the squirrels begin running. Remember to include units.

Solution: Aladar's speed at time $t = 1$ is

$$\sqrt{\left(\left.\frac{dx}{dt}\right|_{t=1}\right)^2 + \left(\left.\frac{dy}{dt}\right|_{t=1}\right)^2} = \sqrt{(4)^2 + (2)^2}$$
$$= \sqrt{20}\ \text{m/s}$$

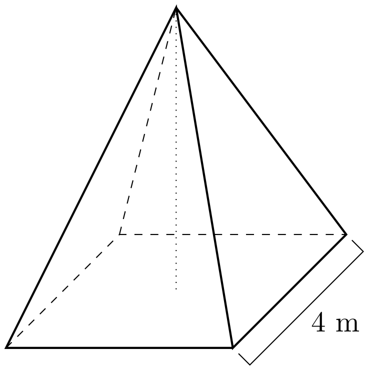
b. [4 points] Find the x- and y-coordinates of the point(s) where their **paths** intersect, if any.

Solution: The paths intersect if the two x-coordinates and two y-coordinates are equal at (possibly different) time values. That is, solutions to the system of equations

$$t = 4s$$
$$t - 3 = s^2$$

Subsituting $t = 4s$ into the second equation gives $s^2 - 4s + 3 = 0$, so $s = 1, 3$. Plugging in, we get the two possible intersection points (4,1) and (12,9). However, Zini would be at the point (12,9) when $t = 12$, which is outside of the domain, thus the only intersection of their paths is at the point (4,1).

12. [9 points] An oil tank has the shape of a pyramid with a square base of side length 4 meters and height 10 meters. The top of the pyramid lies directly above the center of the base. **Be sure to include units in your answers**. Recall that the gravitational constant is $g = 9.8$ m/s^2.



The tank is filled with oil up to a height of 6 meters.

a. [4 points] Write an expression approximating the mass of a thin horizontal slice of thickness Δy located y meters **below** the top of the tank. The density of the oil is 880 kg/m^3. Don't forget to include units.

Solution: Using similar triangles, the side length of a horizontal slice of thickness Δy located y meters below the top of the tank is $\frac{2y}{5}$ meters. Thus the mass of the slice is $(880)\left(\frac{2y}{5}\right)^2 \Delta y$ kg.

b. [5 points] Write a definite integral that represents the total amount of work required to pump all of the oil to the top of the tank. Do not evaluate the integral. Don't forget to include units.

Solution: The total amount of work is

$$\int_4^{10} (880)\left(\frac{2y}{5}\right)^2 (9.8)(y)\, dy \text{ J}.$$

13. [10 points] Suppose a_n and b_n are sequences with the following properties.

- $\sum_{n=1}^{\infty} a_n$ converges.
- $n \le b_n \le e^n$.

For each of the following statements, decide whether the statement is always true, sometimes true, or never true. Circle your answer. No justification is necessary. **You only need to answer 5 of the 7 questions**. Only answer the 5 questions you want graded. If it is unclear which 5 questions are being answered, the first 5 questions you answer will be graded.

a. [2 points] The sequence $\frac{1}{b_n}$ diverges.

ALWAYS **SOMETIMES** **$\boxed{\text{NEVER}}$**

b. [2 points] The sequence a_n is bounded.

$\boxed{\text{ALWAYS}}$ **SOMETIMES** **NEVER**

c. [2 points] The series $\sum_{n=1}^{\infty} \frac{1}{b_n}$ diverges.

ALWAYS **$\boxed{\text{SOMETIMES}}$** **NEVER**

d. [2 points] The series $\sum_{n=1}^{\infty} e^{-a_n}$ converges.

ALWAYS **SOMETIMES** **$\boxed{\text{NEVER}}$**

e. [2 points] The series $\sum_{n=1}^{\infty} a_n^2$ diverges.

ALWAYS **$\boxed{\text{SOMETIMES}}$** **NEVER**

f. [2 points] The series $\sum_{n=1}^{\infty} a_n b_n$ converges.

ALWAYS **$\boxed{\text{SOMETIMES}}$** **NEVER**

g. [2 points] The series $\sum_{n=1}^{\infty} \frac{b_n}{n!}$ converges.

$\boxed{\text{ALWAYS}}$ **SOMETIMES** **NEVER**

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$