Math 116 — Midterm Exam

November 14, 2016

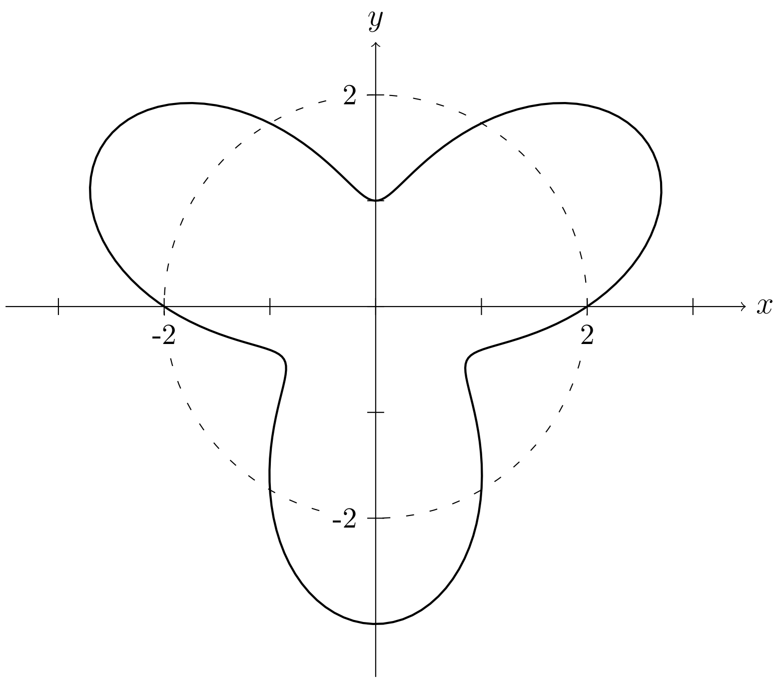
UMID: EXAM SOLUTIONS Initials: ____________

Instructor: ____________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	9	
2	5	
3	5	
4	12	
5	7	
6	5	
7	9	
8	12	
9	10	
10	14	
11	12	
Total	100	

1. [9 points] While assembling a large fan as a holiday gift, you discover the following lovely diagram in the instruction manual. The propeller and base of the fan are illustrated by the polar curve $r = 2 + \sin(3\theta)$ and the circle $r = 2$, respectively.



a. [5 points] Write down, but do **not** evaluate, an integral that gives the arc length of the part of the propeller **outside** the circle in the **first quadrant**.

Solution: The arclength is given by

$$\int_0^{\pi/3} \sqrt{(2+\sin(3\theta))^2 + (3\cos(3\theta))^2}\, d\theta,$$

or alternatively

$$\int_0^{\pi/3} \sqrt{(3\cos(3\theta)\cos(\theta) - (2+\sin(3\theta))\sin(\theta))^2 + (3\cos(3\theta)\sin(\theta) + (2+\sin(3\theta))\cos(\theta))^2}\, d\theta.$$

b. [4 points] Write down the Cartesian equation of the tangent line to the **propeller** at the point $(x, y) = (2, 0)$.

Solution: The tangent line is given by

$$y = \frac{2}{3}(x-2).$$

2. [5 points] Find constants A and B so that the function $h(w)$, defined for $w > 0$ by

$$h(w) = Aw^3 + \frac{1}{w},$$

is a solution to the differential equation

$$w^2 \frac{dh}{dw} - 3wh + B = 0$$

satisfying $h(1) = \frac{3}{2}$. Show all your work, and write your final answers in the spaces provided.

Solution:

$A =$ 1/2

$B =$ 4

3. [5 points] In a recent presidential election between candidate A and candidate B, Shamcorp's rival company Hawk-I tried fixing the election by changing the votes on some of the ballots. For the last three hours of the election (between 5pm and 8pm), the company gained access to the huge ballot box containing 100 million ballots.

Hawk-I employees removed ballots from the ballot box continuously at a rate of 4 million ballots per hour. Those ballots were removed in proportion to the current ratio in the box. Hawk-I employees then instantly changed the the ballots voting for candidate B to vote for candidate A (leaving any votes for candidate A unchanged) before immediately returning the ballots to the box.

Assume that the ballot box always contains 100 million votes, and that the ballot box only contains votes for candidates A and B.

Write a differential equation that models $a(t)$, the number of ballots voting for candidate A, **in millions**, in the ballot box t hours after Hawk-I began changing votes.

Solution:

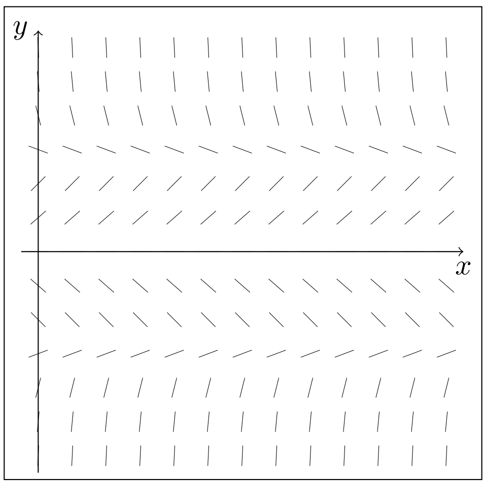
$$\frac{da}{dt} = 4 - \frac{a}{25}$$

4. [12 points] Each graph below is a slope field for one of the differential equations. Beneath each slope field, write the letter of the differential equation in the blank. There is only one correct differential equation for each slope field. Assume $A > 0$ and $B < 0$ are constants. You do not need to show your work. Any ambiguous answers will be marked incorrect.

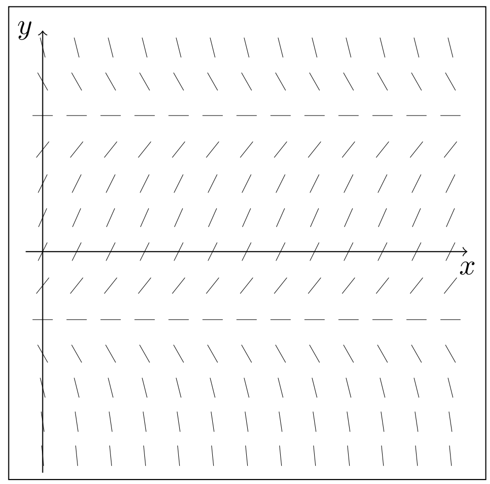
(N) $\dfrac{dy}{dx} = (A-y)^2(B-y)$ (P) $\dfrac{dy}{dx} = (A+y)(B+y)^2$ (Q) $\dfrac{dy}{dx} = (A-y)(B-y)$

(R) $\dfrac{dy}{dx} = (y-A)(B-y)$ (S) $\dfrac{dy}{dx} = (y-A)^2(y+B)$ (T) $\dfrac{dy}{dx} = y(A-y^2)$

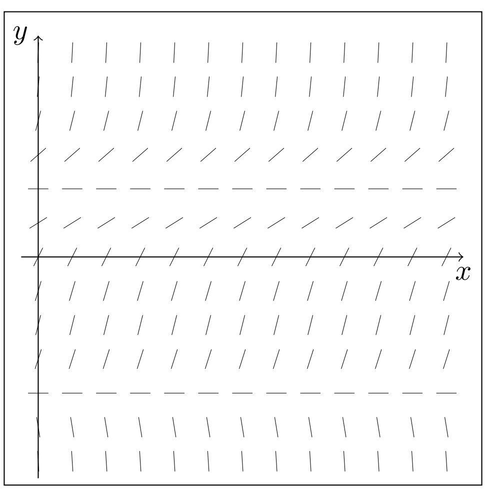
(V) $\dfrac{dy}{dx} = y(y^2+B)$ (W) $\dfrac{dy}{dx} = y(y^2-B)$ (Z) $\dfrac{dy}{dx} = y^2(y+A)$



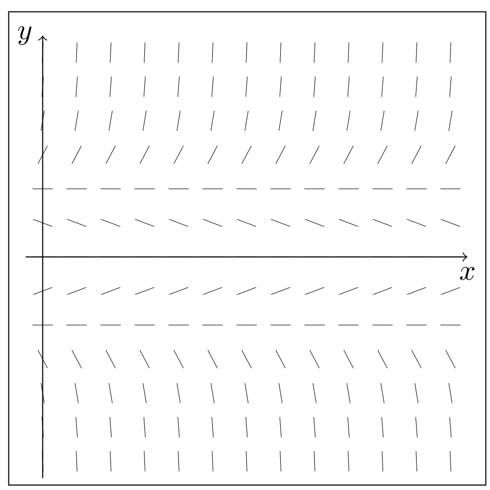
Answer: T



Answer: R



Answer: P



Answer: V

5. [7 points] The Intern has designed an experiment to stabilize the highly radioactive compound Porcinate. In his experimental setup, the amount $P(t)$ of Porcinate in moles, t hours after the experiment began, should satisfy the differential equation

$$\frac{dP}{dt} - \frac{tP}{\ln(P)} = 0.$$

Use separation of variables to find a solution $P(t)$ satisfying $P(3) = e$.

Solution: We have

$$\int \frac{\ln(P)}{P}\, dP = \int t\, dt,$$

so

$$\frac{(\ln(P))^2}{2} = \frac{t^2}{2} + C$$

for some constant C. The initial condition $P(3) = e$ gives

$$\frac{(\ln(e))^2}{2} = \frac{3^2}{2} + C,$$

so $C = -4$. Hence

$$P(t) = e^{\sqrt{t^2-8}}.$$

6. [5 points] The Intern is also studying a compound called Bovinate. The amount $B(t)$ of Bovinate in moles, t hours after an experiment began, satisfies the differential equation

$$\frac{dB}{dt} = 2B(1-B)(t+B)^2.$$

a. [3 points] List all equilibrium solutions of the differential equation. Indicate whether each is stable or unstable.

Solution: There are two equilibria: $B = 0$ (unstable) and $B = 1$ (stable).

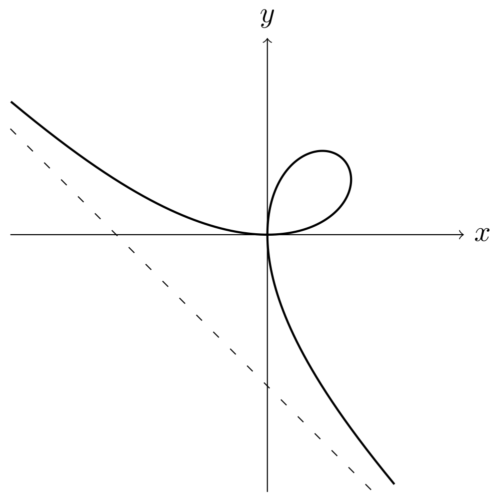
b. [2 points] If the initial amount of Bovinate were 0.5 moles, what would happen to the amount of Bovinate in the long run?

Solution: The amount of Bovinate would approach 1 mole asymptotically from below.

7. [9 points] For $-\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, consider the polar curve

$$r = \frac{\sin(2\theta)}{\cos(\theta) + \sin(\theta)}.$$

The curve has an asymptote, the dashed line in the picture, as θ approaches $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$.



a. [4 points] Write down, but do **not** evaluate, an integral that gives the area inside the loop in the first quadrant.

Solution: The area is given by

$$\frac{1}{2}\int_0^{\pi/2} \left(\frac{\sin(2\theta)}{\cos(\theta) + \sin(\theta)}\right)^2 d\theta.$$

b. [2 points] Find a formula for the quantity $x + y$ in terms of the variable θ. Write your answer in the space provided.

Solution:

$$x + y = \frac{\sin(2\theta)}{\cos(\theta) + \sin(\theta)}(\cos(\theta) + \sin(\theta)) = \sin(2\theta)$$

c. [2 points] Find the limit of $x + y$ as $\theta \to \left(\frac{3\pi}{4}\right)^-$. No justification is needed.

Solution: The specified limit is

$$\lim_{\theta \to (3\pi/4)^-} \sin(2\theta) = \sin\left(\frac{3\pi}{2}\right) = -1.$$

d. [1 point] Write down the Cartesian equation of the asymptote. No justification is needed.

Solution: The asymptote is given by $x + y = -1$.

8. [12 points] For each of the following parts, circle the correct answer. Ambiguous answers will receive no credit. You do **not** need to show your work.

a. [3 points] For nonzero constants a and b, the curve $r = \dfrac{a}{\sin(\theta) + b\cos(\theta)}$ is a line. What is the Cartesian equation of the line?

$\boxed{y = -bx + a}$ $\quad y = ax - b \quad y = bx - a \quad y = -ax + b \quad y = -bx - a$

b. [3 points] Raymond Green left a bowl of ice cream in a 50°C sauna. Over the first $2\ln(2)$ hours, the ice cream goes from -10°C to 20°C. Which of the following describes the change in $Q(t)$, the temperature of the ice cream in °C after t hours?

$$\frac{dQ}{dt} = \frac{Q-50}{2} \qquad \frac{dQ}{dt} = 2(50-Q) \qquad \frac{dQ}{dt} = \ln(2)\left(25 - \frac{Q}{2}\right)$$

$$\boxed{\frac{dQ}{dt} = 25 - \frac{Q}{2}} \qquad \frac{dQ}{dt} = -\frac{1}{2}(Q-25)$$

c. [3 points] Let $\alpha > 0$ be a constant. What is the value of $\displaystyle\lim_{u\to\infty}\left(\frac{u}{u-\alpha}\right)^{u-\alpha}$?

$\boxed{e^{\alpha}}$ $\quad 1 \quad$ DIVERGES $\quad e^{1/\alpha} \quad \alpha$

d. [3 points] Consider the differential equation $y' = 1 + \beta xy$, where β is a constant, and let $y(x)$ be a solution satisfying $y(0) = 1$. For which value of β does Euler's method with 2 steps give the estimate $y(4) \approx 0$?

$$-\frac{3}{4} \qquad -\frac{1}{6} \qquad \boxed{-\frac{5}{12}} \qquad -\frac{1}{2} \qquad -\frac{6}{11}$$

9. [10 points] Suppose that f is function with the following properties:

f is differentiable $\qquad f(x) > 0$ for all x $\qquad \displaystyle\int_1^\infty f(x)\,dx$ converges.

For each of the following parts, determine whether the statement is always, sometimes, or never true by circling the appropriate answer. No justification is needed.

a. [2 points] $\displaystyle\int_{500}^\infty 1000f(x)\,dx$ converges.

$\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

b. [2 points] $\displaystyle\int_1^\infty (f(x))^{2/3}\,dx$ converges.

ALWAYS $\boxed{\textbf{SOMETIMES}}$ **NEVER**

c. [2 points] $\displaystyle\int_1^\infty (f(x))^{3/2}\,dx$ converges.

ALWAYS $\boxed{\textbf{SOMETIMES}}$ **NEVER**

d. [2 points] $\displaystyle\int_0^1 f\left(\frac{1}{x}\right) dx$ converges.

$\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

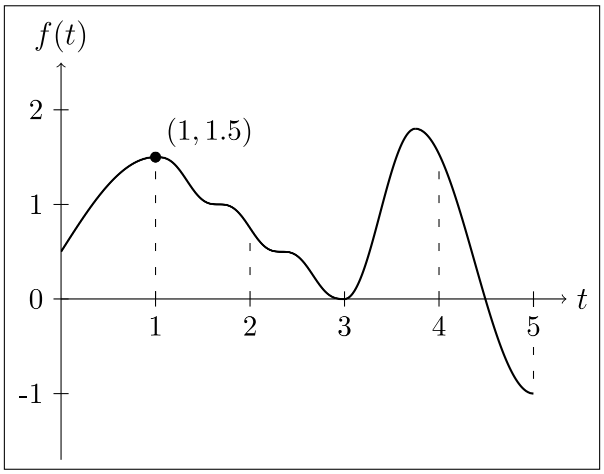
e. [2 points] $\displaystyle\int_1^\infty \frac{f'(x)}{f(x)}\,dx$ converges. $\qquad \left(\text{Note: } \dfrac{f'(x)}{f(x)} = \dfrac{d}{dx}\ln(f(x)).\right)$

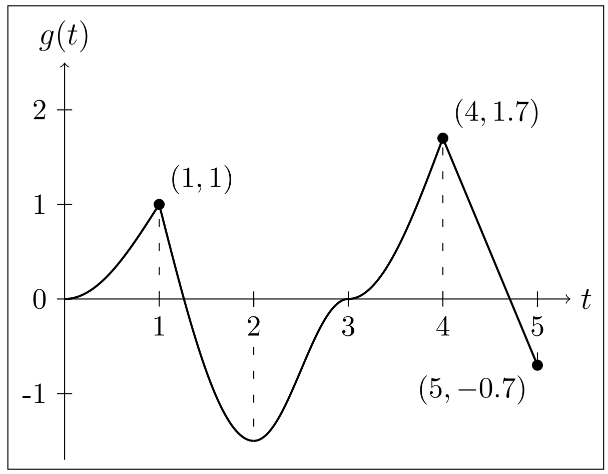
ALWAYS **SOMETIMES** $\boxed{\textbf{NEVER}}$

10. [14 points] Fearing that she is losing authority over her robot ward, Dr. Durant has installed a tracking chip in Steph's mainframe. The chip gives Steph's location separately in x- and y-coordinates, where the units of the axes are miles, Dr. Durant's office corresponds to the origin $(x, y) = (0, 0)$, the positive y-axis points north, and the positive x-axis points east. On night 1, Dr. Durant noticed unusual levels of activity; t hours after midnight, Steph began moving according to the parametric equations

$$x = f(t) \qquad\qquad y = g(t),$$

where $f(t)$ and $g(t)$ are plotted below for $0 \le t \le 5$.





a. [2 points] When was Steph farthest north and south on night 1? Write your answers in the blanks provided. You do **not** need to show your work.

Solution: **North:** 4 **a.m.** **South:** 2 **a.m.**

b. [3 points] What was Steph's speed at $t = 4.9$ on night 1? You may use the fact that $f'(4.9) = -1$. Include units.

Solution: Her speed is $\sqrt{(-1)^2 + (-2.4)^2} = \sqrt{6.76}$ mi/hr.

c. [2 points] What direction was Steph moving at $t = 2$ on night 1? Circle only one answer.

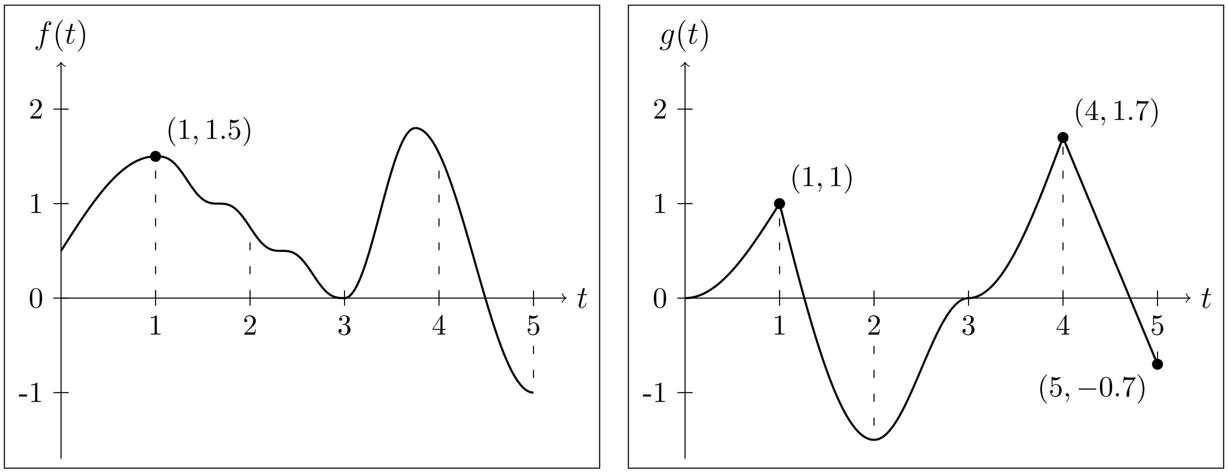
NORTH AND EAST **EAST ONLY** **SOUTH AND EAST**

NORTH AND WEST $\boxed{\textbf{WEST ONLY}}$ **SOUTH AND WEST**

10 (continued). Recall that on night 1, Steph's position was given by the parametric equations

$$x = f(t) \qquad\qquad y = g(t),$$

where $f(t)$ and $g(t)$ are plotted below for $0 \le t \le 5$. As before, Dr. Durant's office is at the origin $(x, y) = (0, 0)$, the positive y-axis points north, and the positive x-axis points east.



d. [3 points] How far away was Steph from Dr. Durant's office at $t = 1$ on night 1?

Solution: Steph was $\sqrt{1^2 + 1.5^2} = \sqrt{3.25}$ mi away.

On night 2, Steph's movements were even stranger, following the parametric equations

$$x = \int_0^t f(s)\,ds \qquad\qquad y = \int_0^t g(s)\,ds.$$

e. [2 points] What direction was Steph moving at $t = 2$ on night 2? Circle only one answer.

NORTH AND EAST **EAST ONLY** $\boxed{\textbf{SOUTH AND EAST}}$

NORTH AND WEST **WEST ONLY** **SOUTH AND WEST**

f. [2 points] Did Steph come to a stop between midnight and 5 a.m. on night 2? If so, at what time(s) did she come to a stop?

Solution: Yes; she came to a stop at 3 a.m.

11. [12 points] Determine whether the following integrals converge or diverge. If an integral converges, find its exact value (i.e., no decimal approximations) and write it in the blank provided. If it diverges, circle "DIVERGES" and explain why. In any case, **show all your work, indicating any theorems you use, and using proper syntax and notation**.

a. [6 points] $\displaystyle\int_0^\infty 2xe^{-cx}\,dx$, where $c > 0$ is a constant

DIVERGES **CONVERGES TO** $2/c^2$

Solution: For $b \geq 0$ we have

$$\int_0^b xe^{-cx}\,dx = -\frac{xe^{-cx}}{c}\bigg|_0^b + \frac{1}{c}\int_0^b e^{-cx}\,dx = -\frac{be^{-cb}}{c} + \frac{1-e^{-cb}}{c^2},$$

so

$$\int_0^\infty 2xe^{-cx}\,dx = 2\lim_{b\to\infty}\left(-\frac{be^{-cb}}{c} + \frac{1-e^{-cb}}{c^2}\right) = \frac{2}{c^2}.$$

b. [6 points] $\displaystyle\int_0^1 \frac{x}{\sqrt{x^5+x^7}}\,dx$

$\boxed{\textbf{DIVERGES}}$ **CONVERGES TO** __________________

Solution: For $0 < x \leq 1$ we have

$$\frac{x}{\sqrt{x^5+x^7}} \geq \frac{x}{\sqrt{2x^5}} = \frac{1}{\sqrt{2}x^{3/2}}.$$

Since

$$\frac{1}{\sqrt{2}}\int_0^1 \frac{dx}{x^{3/2}}$$

diverges by the p-Test with $p = \frac{3}{2}$, the original integral diverges by comparison.

Alternatively, notice that

$$\lim_{x\to 0^+}\frac{x/\sqrt{x^5+x^7}}{1/x^{3/2}} = 1.$$

Since

$$\int_0^1 \frac{dx}{x^{3/2}}$$

diverges by the p-Test with $p = \frac{3}{2}$, the original integral diverges by the Limit Comparison Test.