Math 116 — Final Exam

December 19, 2016

UMID: EXAM SOLUTIONS Initials: ______

Instructor: ______ Section: ______

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 13 pages including this cover. There are 12 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	4	
2	5	
3	8	
4	5	
5	5	
6	10	
7	7	
8	11	
9	12	
10	14	
11	10	
12	9	
Total	100	

1. [4 points] Suppose that the power series $\sum_{n=0}^{\infty} c_n(x-3)^n$ converges at $x = 6$ and diverges at $x = -2$. What can you say about the behavior of the power series at the following values of x? For each part, circle the correct answer. Ambiguous responses will be marked incorrect.

a. [1 point] At $x = -3$, the power series...

CONVERGES $\boxed{\textbf{DIVERGES}}$ **CANNOT DETERMINE**

b. [1 point] At $x = 0$, the power series...

CONVERGES **DIVERGES** $\boxed{\textbf{CANNOT DETERMINE}}$

c. [1 point] At $x = 8$, the power series...

CONVERGES **DIVERGES** $\boxed{\textbf{CANNOT DETERMINE}}$

d. [1 point] At $x = 2$, the power series...

$\boxed{\textbf{CONVERGES}}$ **DIVERGES** **CANNOT DETERMINE**

2. [5 points] Determine the **radius** of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^{2n}.$$

Justify your work carefully and write your final answer in the space provided. Limit syntax will be enforced.

Solution: For $n = 0, 1, \ldots$, let $a_n = \frac{(2n)!}{(n!)^2}$. We have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!} = \frac{(2n+1)(2n+2)}{(n+1)^2} \to 4$$

as $n \to \infty$. Hence the radius of convergence is $\sqrt{\frac{1}{4}} = \frac{1}{2}$.

Radius of convergence = $\frac{1}{2}$

3. [8 points] For $n = 1, 2, 3, \ldots$ consider the sequence a_n given by

$$a_n = \frac{-1}{2^{(n+1)/2}} \text{ if } n \text{ is odd,} \qquad a_n = \frac{1}{3^{n/2}} \text{ if } n \text{ is even.}$$

a. [2 points] Write out the first 5 terms of the sequence a_n.

Solution: The first five terms are

$$-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{8}.$$

b. [2 points] The series $\displaystyle\sum_{n=1}^{\infty} a_n$ is alternating. In a sentence or two, explain why the Alternating Series Test **cannot** be used to determine whether $\displaystyle\sum_{n=1}^{\infty} a_n$ converges or diverges.

Solution: The condition $|a_{n+1}| < |a_n|$ does not hold for all n. (It does not even hold eventually.)

c. [4 points] The series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Show that it converges, either by using theorems about series, or by computing its exact value.

Solution: One possible answer is that the series is equal to the difference of two convergent geometric series:

$$\sum_{k=1}^{\infty} \frac{1}{3^k} - \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} - \frac{\frac{1}{2}}{1 - \frac{1}{2}} = -\frac{1}{2}.$$

Another answer uses the Comparison Test; for $n = 1, 2, \ldots$, let $b_n = \frac{1}{n^2}$, and notice that $|a_n| \le b_n$ eventually. Since $\sum_{n=1}^{\infty} b_n$ converges by the p-Test ($p = 2$), $\sum_{n=1}^{\infty} |a_n|$ converges by comparison. Hence the original series converges.

4. [5 points] The following series **diverges**:

$$\sum_{n=2}^{\infty} \frac{n}{n^2 + \ln(n)}.$$

Use theorems about infinite series to **show** that the series diverges. Give full justification, showing all your work and indicating any theorems or tests that you use.

Solution: One solution uses the Comparison Test. Notice that

$$\frac{n}{n^2+\ln(n)} \geq \frac{n}{n^2+n^2} = \frac{1}{2n}$$

for all $n \geq 2$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges by the p-Test ($p = 1$), the original series diverges by comparison.

Alternatively, let $a_n = \dfrac{n}{n^2 + \ln(n)}$ and $b_n = \dfrac{1}{n}$ for all $n \geq 2$, and notice that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

Since $\displaystyle\sum_{n=2}^{\infty} b_n$ converges by the p-Test ($p = 1$), the original series diverges by the Limit Comparison Test.

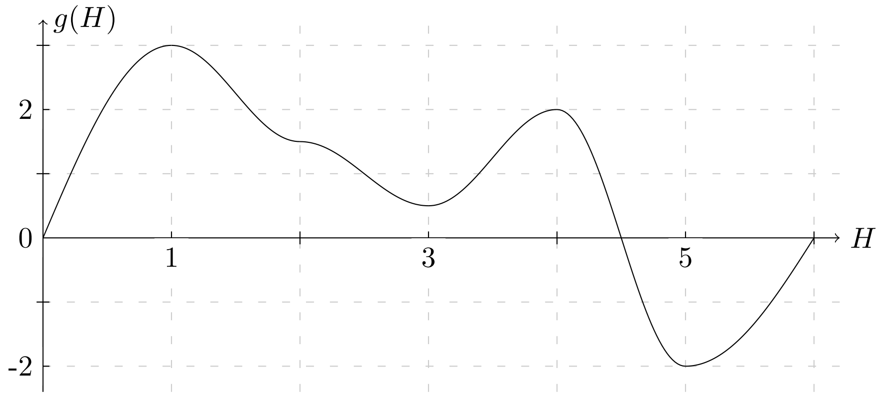
5. [5 points] Let $\alpha > 0$ be a constant. Compute the first 3 terms of the Taylor series of $f(x) = \dfrac{x}{\sqrt{1+\alpha x}}$ about $x = 0$. Write the appropriate coefficients in the spaces provided.

$$\underline{\quad 0 \quad} + \underline{\quad 1 \quad}\, x + \underline{\quad -\frac{\alpha}{2} \quad}\, x^2 + \cdots$$

6. [10 points] After receiving a termination notice, The Intern has begun to read up on the global job market. A dubious pop-economics book he is reading claims that the rate at which interns are hired or terminated in a large company is purely a function of the number of interns at the company. Specifically, it states that

$$\frac{dH}{dt} = g'(H),$$

where $H(t)$ gives the number of interns at a company, in thousands, after t days, and $g(H)$ is a differentiable function. A graph of $g(H)$ (**not** $g'(H)$) is given in the book:



a. [2 points] What are the units of $g'(H)$?

Solution: The units are thousands of interns per day.

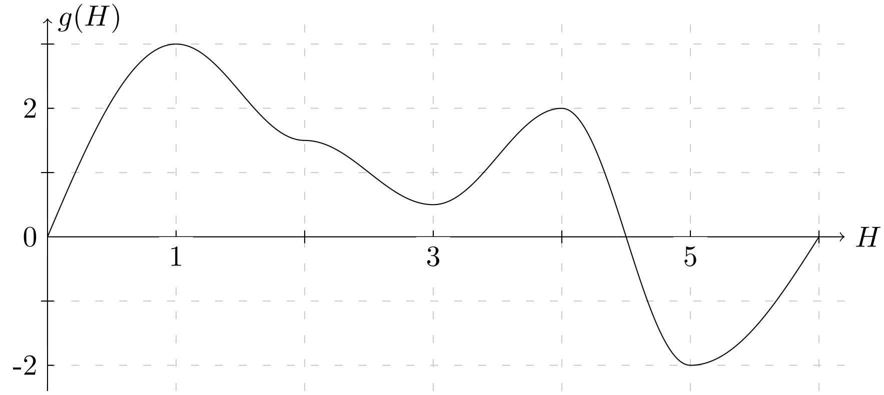
b. [3 points] Are there any **stable** equilibrium solutions of the differential equation? If so, what are they?

Solution: Yes; the stable equilibrium solutions are $H = 1$ and $H = 4$.

6. (continued). Recall that the number of interns in thousands $H(t)$ satisfies

$$\frac{dH}{dt} = g'(H),$$

where a graph of $g(H)$ (**not** $g'(H)$) is given below:



c. [2 points] If a company starts with 3,500 interns, what will happen to the number of interns in the long run?

Solution: The number of interns will approach 4,000 asymptotically from below.

d. [1 point] Estimate the number of interns at which the number of interns is decreasing the fastest.

Solution: The number of interns is decreasing the fastest when there are $\approx 4,500$ interns.

e. [2 points] Suppose that a company begins with 5,500 interns. If you used Euler's method to estimate how many interns there will be 5 days from now, would you expect an underestimate or an overestimate? Justify your answer briefly.

Solution: The corresponding solution of the differential equation is concave up, so we expect Euler's method to yield an underestimate.

7. [7 points] The *Legendre equation* is a differential equation that arises in the quantum mechanical study of the hydrogen atom. In one of its forms, the Legendre equation is

$$(1-x^2)y'' - 2xy' + 12y = 0.$$

For this problem, let y be a solution to the Legendre equation satisfying $y(\frac{1}{2}) = 2$ and $y'(\frac{1}{2}) = 3$. Assume that the Taylor series for $y(x)$ about $x = \frac{1}{2}$ converges to $y(x)$ for all $-\frac{1}{2} < x < \frac{3}{2}$.

a. [4 points] In the blank below, write down $P_2(x)$, the degree 2 Taylor polynomial of $y(x)$ near $x = \frac{1}{2}$. Your answer should not contain the function $y(x)$ or any of its derivatives.

$$P_2(x) = \underline{\quad 2 + 3\left(x - \frac{1}{2}\right) - 14\left(x - \frac{1}{2}\right)^2 \quad}$$

b. [3 points] Compute the limit

$$\lim_{x \to 1/2} \frac{y(x) - \frac{1}{2} - 3x}{(x - \frac{1}{2})^2}.$$

Solution: The limit is -14.

8. [11 points] In this problem, we consider the parametric curve given by

$$x = f(t) \qquad\qquad y = g(t)$$

for all t, where f and g are twice-differentiable functions. Some values of f and g and their derivatives are given in the tables below.

t	1	2	3	4	5
$f(t)$	-3	-4	-3	-1	1
$g(t)$	5	2	-2	-4	-1

t	1	2	3	4	5
$f'(t)$	-2	0	1	3	1
$g'(t)$	-4	-2	-1	0	2

a. [1 point] In the space provided, write an integral that gives the arc length of the parametric curve from $t = 1$ to $t = 5$.

Arc length = $\int_1^5 \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$

b. [3 points] Use a midpoint sum with as many subdivisions as possible to estimate your integral from part **a**. Write out all the terms in your sum, and do **not** simplify.

Solution: The midpoint sum is $2(\sqrt{0^2 + (-2)^2} + \sqrt{3^2 + 0^2})$.

c. [3 points] Find the Cartesian equation for the tangent line to the parametric curve in the xy-plane at $t = 1$.

Solution: In point-slope form, the tangent line is given by $y - 5 = 2(x + 3)$.

d. [2 points] Consider the tangent lines to the parametric curve at the t-values $t = 1, 2, 3, 4, 5$. Are any of these lines **perpendicular** to each other? If so, list any **two** t-values for which the tangent lines are perpendicular. If not, write "NO."

Solution: The tangent lines corresponding to $t = 2$ and $t = 4$ are perpendicular.

e. [2 points] As t ranges from 1 to 5, the corresponding part of the parametric curve intersects the line $y = x$ exactly once. Which interval contains the t-value for which the curve intersects the line $y = x$? Circle your answer. You do not need to show any work.

$(1, 2)$ $\qquad$ $(2, 3)$ $\qquad$ $\boxed{(3, 4)}$ $\qquad$ $(4, 5)$

9. [12 points] Read the following parts carefully, and circle the appropriate answer(s). Some parts may have more than one correct answer.

a. [3 points] Circle the value(s) of x for which the following identity holds:

$$2 = x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots$$

$\sqrt[3]{\ln(2)}$ $\boxed{\sqrt[3]{\ln(3)}}$ $(\ln(2))^3$

e^{2^3} $\ln(\sqrt[3]{3})$ $e^{\sqrt[3]{3}}$

b. [3 points] Raymond Green's pet anaconda Sheela grew 5 m in length over the past month. The veterinarian says that each month, the increase in Sheela's length will be 40% of the increase the month before. How much longer (in meters) will Sheela be one year from now? Circle **all** that apply.

$\displaystyle\sum_{k=0}^{12} 5(0.4)^k$ $\boxed{\dfrac{2(1-(0.4)^{12})}{1-0.4}}$ $\boxed{\displaystyle\sum_{k=1}^{12} 5(0.4)^k}$

$\dfrac{5(1-(0.4)^{12})}{1-0.4}$ $\dfrac{2(1-(0.4)^{13})}{1-0.4}$

c. [3 points] Let $\alpha > 0$ be a constant. What is the value of the convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\alpha^n}{(2n)!}$?

$\cos(\alpha) - 1$ $\cos(\sqrt{\alpha}) - 1$ $1 - \cos(\alpha)$

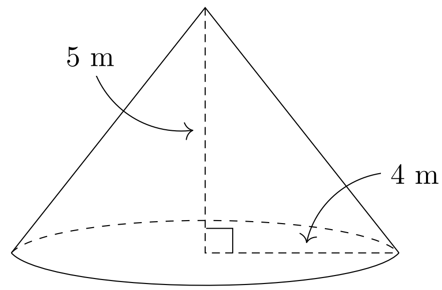
$\alpha - \cos(\sqrt{\alpha})$ $\boxed{1 - \cos(\sqrt{\alpha})}$ $\cos(\alpha) - \alpha$

d. [3 points] Which of the following series converge **absolutely**? Circle **all** that apply.

$\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{\sin^{99}(n)}{n^2}}$ $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ $\displaystyle\sum_{n=2}^{\infty} \frac{8^n + (-1)^n 10^n}{9^n}$

$\boxed{\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n(\ln(n))^{1.01}}}$ $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n}$

10. [14 points] The conical tank pictured below has a base of radius 4 meters and a height of 5 meters. It is filled to the top with a toxic liquid, PGM, which has a constant density of 1000kg/m^3. The gravitational constant is $g = 9.8$ m/s^2.



a. [4 points] Write an expression for the volume of a circular slice of thickness Δh, a distance h meters from the base.

Solution: The volume is $\pi(4 - \frac{4}{5}h)^2\,\Delta h$ m^3.

b. [3 points] Dr. Durant is trying to take over Shamcorp by making a device using PGM. Dr. Durant must pump all the PGM to the top of the cone to complete his device. Write an expression involving integrals which gives the work Dr. Durant does pumping all of the liquid to the top of the container.

Solution: The work done is

$$9.8\pi \int_0^5 \left(4 - \frac{4}{5}h\right)^2 (5 - h)\, dh \quad \text{kJ}.$$

c. [3 points] Write an expression **involving integrals** for A, the average radius of a circular slice of the cone for $0 \le h \le 5$.

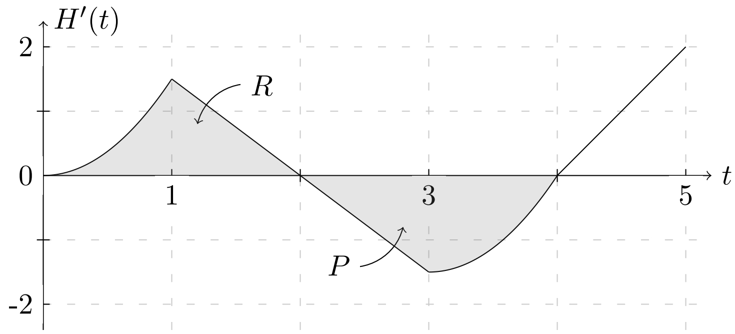
Solution: We have

$$A = \frac{1}{5}\int_0^5 \left(4 - \frac{4}{5}h\right) dh \quad \text{m}.$$

d. [4 points] Raymond Green is also building a PGM device, but he has a cylindrical container, oriented so that its circular base is on the ground, filled to the top with PGM. The container has a height of 5 meters and has radius A (the average radius of Dr. Durant's container). Whoever does the least amount of work pumping the PGM to the top of their container will rule Shamcorp. Will it be Dr. Durant or Raymond Green? Give a brief justification for your answer.

Solution: The volume of Durant's container is greater than that of Green's container; moreover, the center of mass of Durant's container is closer to the ground than that of Green's container. This means that Durant will do more work, so Green will rule ShamCorp.

11. [10 points] Perhaps sensing that the end is near, Steph is preparing her eventual legacy. While experimenting with an accelerant called Equinate, Steph found that she could alter its *heat of combustion* by aging the Equinate in barrels. Let $H(t)$ be the heat of combustion of Equinate, measured in hundreds of millions of Joules per kilogram, after the Equinate has been aged for t years. A graph of the derivative $H'(t)$ is below; note that $H'(t)$ is linear for $1 < t < 3$ and $4 < t < 5$. Let $R > 0$ be the area of the region between the t-axis and the graph of $H'(t)$ for $0 \le t \le 2$. Let $P > 0$ be the corresponding area for $2 \le t \le 4$.



a. [3 points] The heat of combustion of Equinate, after aging for 5 years, is 200 million J/kg. What is the heat of combustion of Equinate that has not been aged at all? Your answer may include R and P.

Solution: The answer is $1 - R + P$ hundred million J/kg.

b. [3 points] Steph is storing four barrels of Equinate in the ShamCorp basement. Barrel A has not been aged; barrel B has been aged for 2 years; barrel C has been aged for 4 years; and barrel D has been aged for 5 years. In the spaces provided, list the barrels A, B, C, and D in **increasing** order of heat of combustion.

C < A < D < B

c. [4 points] At some time between 4 and 5 years of aging, the heat of combustion of Equinate is the same as if it had not been aged at all. After how many years of aging does this occur? Your answer may include R and P.

Solution: This occurs after $4 + \sqrt{P - R}$ years of aging.

12. [9 points] Three intervals are given below. In the space next to each interval, write the letter(s) corresponding to each power series (A)-(I) (below) whose interval of convergence is **exactly** that interval. There may be more than one answer for each interval. If there are intervals below for which none of the power series (A)-(I) converge on that interval, write "NONE" in the space next to the interval. You do **not** need to show your work.

a. [3 points] $(-2, 2)$: B

b. [3 points] $(0, 10]$: C

c. [3 points] $[0, \infty)$: NONE

(A) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{4n+2}}{n!}$ (B) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n(2x)^n}{4^n}$ (C) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^n}{n5^n}$

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{\sqrt{n}}$ (E) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (F) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$

(G) $\displaystyle\sum_{n=1}^{\infty} \frac{(\frac{1}{2}x)^n}{n^2}$ (H) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-5)^n}{5^n}$ (I) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n}$

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$