Math 116 — Second Midterm — November 13, 2017

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	9	
3	12	
4	13	
5	5	

Problem	Points	Score
6	7	
7	12	
8	9	
9	12	
10	9	
Total	100	

1. [12 points] Consider the infinite sequences c_n, d_n, j_n and ℓ_n, defined for $n \geq 1$ as follows:

$$c_n = \sum_{k=1}^{n} \frac{(-1)^k}{k!}$$
$$d_n = \arctan(1.1^n)$$
$$j_n = \int_0^{n^3} e^{2x}\, dx$$
$$\ell_n = \sin(x^n) \text{ for some fixed value of } x \text{ satisfying } 0 < x < 1.$$

a. [8 points] Decide whether each of these sequences is bounded, unbounded, always increasing, and/or always decreasing. Record your conclusions by clearly circling the correct descriptions below. Contradictory conclusions will be marked incorrect.

i. The sequence c_n is

$\boxed{\text{bounded}}$ unbounded increasing decreasing

ii. The sequence d_n is

$\boxed{\text{bounded}}$ unbounded $\boxed{\text{increasing}}$ decreasing

iii. The sequence j_n is

bounded $\boxed{\text{unbounded}}$ $\boxed{\text{increasing}}$ decreasing

iv. The sequence ℓ_n is

$\boxed{\text{bounded}}$ unbounded increasing $\boxed{\text{decreasing}}$

b. [4 points]
For parts i and ii below, decide whether the sequence converges or diverges.

- If the sequence converges, circle "converges", find the value to which it converges, and write this value on the answer blank provided.
- If the sequence diverges, circle "diverges".

i. The sequence d_n

$\boxed{\textbf{Converges to } \frac{\pi}{2}}$ **Diverges**

Solution: Note that the geometric sequence 1.1^n diverges to ∞.
So since $\lim_{x\to\infty} \arctan(x) = \pi/2$, we have $\lim_{n\to\infty} \arctan(1.1^n) = \pi/2$.
That is, the sequence d_n converges to $\pi/2$.

ii. The sequence j_n

Converges to ____________ $\boxed{\textbf{Diverges}}$

Solution: The sequence j_n diverges because it is unbounded or, alternatively, because the improper integral $\int_0^{\infty} e^{2x}\, dx$ diverges (e.g. by direct comparison since $e^{2x} \geq 1$ for $x \geq 0$).

2. [9 points] Consider the graph of $y = e^{-14x}$ for $x \geq 0$.

a. [3 points] Let $\mathcal{R}$ be the region in the first quadrant between the graph of $y = e^{-14x}$ and the x-axis. Which of the following improper integrals best expresses the volume of the solid that is obtained by rotating $\mathcal{R}$ around the x-axis?

Circle one:

$\int_0^\infty \pi e^{-14x}\,dx$ $\qquad$ $\int_0^\infty x e^{-14x}\,dx$ $\qquad$ $\int_0^\infty x e^{-28x}\,dx$ $\qquad$ $\boxed{\int_0^\infty \pi e^{-28x}\,dx}$

$\frac{\pi}{7}\int_0^1 \ln(y)\,dy$ $\qquad$ $\frac{\pi}{14}\int_0^1 y\ln(y)\,dy$ $\qquad$ $\frac{1}{14}\int_0^1 y\ln(y^2)\,dy$

Solution: A thin slice of this solid of thickness Δx taken perpendicular to the x-axis at the point x has approximate volume $\pi(e^{-14x})^2 = \pi e^{-28x}$ cubic units. If b is a positive number, the volume of the portion of this solid between $x = 0$ and $x = b$ is $\int_0^b \pi e^{-28x}\,dx$.

So the volume of the entire solid would be $\lim_{b\to\infty}\int_0^b \pi e^{-28x}\,dx = \int_0^\infty \pi e^{-28x}\,dx$.

b. [6 points] Determine whether the improper integral you circled in part **a** converges or diverges.

- If the integral converges, circle "converges", find its exact value (i.e. no decimal approximations), and write the exact value on the answer blank provided.
- If the integral diverges, circle "diverges" and justify your answer.

In either case, **you must show all your work carefully using correct notation**. Any direct evaluation of integrals must be done **without using a calculator.**

Converges to $\underline{\quad \frac{\pi}{28} \quad}$ $\qquad\qquad$ **Diverges**

Solution: We use the definition of this type of improper integral to find its value.

$$\begin{aligned}
\int_0^\infty \pi e^{-28x}\,dx &= \lim_{b\to\infty}\int_0^b \pi e^{-28x}\,dx \\
&= \lim_{b\to\infty}\left[-\frac{\pi}{28}e^{-28x}\right]\Bigg|_{x=0}^{x=b} \\
&= \lim_{b\to\infty}\left[-\frac{\pi}{28}\left(e^{-28b}-1\right)\right] \\
&= \frac{\pi}{28}\ \left(\text{since } \lim_{b\to\infty} e^{-28b} = 0\right).
\end{aligned}$$

So $\int_0^\infty \pi e^{-28x}\,dx$ converges to $\frac{\pi}{28}$.

3. [12 points] Define a sequence a_n for $n \geq 1$ by $a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1/3 & \text{if } n \text{ is even.} \end{cases}$

Note that this means the first three terms of this sequence are $a_1 = 0$, $a_2 = 1/3$, $a_3 = 0$.

a. [2 points] Does the sequence $\{a_n\}$ converge or diverge?

Circle one: **converges** $\boxed{\textbf{diverges}}$

Briefly explain your answer.

Solution: This sequence does not converge because the terms of the sequence alternate forever between 0 and 1/3, and, in particular, never approach a single value. In other words, $\lim_{n\to\infty} a_n$ does not exist.

b. [4 points] Consider the power series $\sum_{n=1}^{\infty} \frac{(3 \cdot a_{2n})}{n3^n} x^n$.

i. Write out the partial sum with three terms for this power series.

Solution: Note that $2n$ is even for all integers n, so a_{2n} is always equal to $1/3$. Therefore, the power series can be rewritten as $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$.

The partial sum with three terms is thus $\frac{x}{3} + \frac{x^2}{2 \cdot 9} + \frac{x^3}{3 \cdot 27} = \frac{x}{3} + \frac{x^2}{18} + \frac{x^3}{81}$.

Answer: $\frac{x}{3} + \frac{x^2}{18} + \frac{x^3}{81}$

ii. Give the exact value of this partial sum when $x = -\frac{1}{2}$.

Solution: Substituting $x = -1/2$ into the partial sum above gives

$$\frac{(-1/2)}{3} + \frac{(-1/2)^2}{18} + \frac{(-1/2)^3}{81} = -\frac{1}{6} + \frac{1}{72} - \frac{-1}{648} = -\frac{25}{162}.$$

Answer: $-\frac{25}{162}$

c. [6 points] Determine the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$.

Show every step of any calculations and fully justify your answer with careful reasoning. Write your final answer on the answer blank provided.

Solution: To compute the radius of convergence we apply the Ratio Test to the general term of the power series, which we will call $b_n = \frac{x^n}{n \cdot 3^n}$.

$$\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = \lim_{n\to\infty} \frac{n3^n}{(n+1)3^{n+1}} \cdot |x| = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{|x|}{3} = \frac{|x|}{3}.$$

By the Ratio Test, this power series will therefore converge if $|x|/3 < 1$, i.e. $|x| < 3$, and diverge if $|x| > 3$. This shows that the radius of convergence is $R = 3$, and the interval of convergence is $(-3, 3)$ together with possibly one or both of the endpoints. We still need to check convergence at the endpoints.

At $x = -3$ the power series reduces to

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cdot \frac{3^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This series converges by the alternating series test. (It is the alternating harmonic series.) This implies that $x = -3$ is in the interval of convergence.

Finally, at $x = 3$, the power series reduces to

$$\sum_{n=1}^{\infty} \frac{(3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{3^n}{3^n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

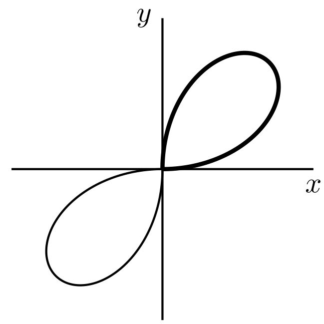
This series diverges by p-test with $p = 1$. (It is the harmonic series.) This implies that $x = 3$ is NOT in the interval of convergence.

Hence, the interval of convergence is $[-3, 3)$.

Answer: $[-3, 3)$

4. [13 points] The orbit of a single electron around the nucleus of an atom is determined by the energy level of that electron and by the other electrons orbiting the nucleus. We can model one electron's orbital in two-dimensions as follows. Suppose that the nucleus of an atom is centered at the origin. Then the (so-called "$2p_1$") orbital has the shape shown below.

This shape is made up of two regions that we call "lobes". The outer edge of the lobes are described by the polar equation $r = k\sin(2\theta)$ for some positive constant k. Note that only the relevant portion of the polar curve $r = k\sin(2\theta)$ is shown.



The "top lobe" is the portion in the first quadrant (shown in bold).

a. [2 points] For what values of θ with $0 \le \theta \le 2\pi$ does the polar curve $r = k\sin(2\theta)$ pass through the origin?

Solution: The curve passes through the origin when $r = 0$, i.e. when $\sin(2\theta) = 0$, or $\boxed{\theta = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ \text{and } 2\pi.}$

b. [3 points] For what values of θ does the polar curve $r = k\sin(2\theta)$ trace out the "top lobe"? Give your answer as an interval of θ values.

Solution: From part (a), we are at the origin when $\theta = 0$; as θ increases from 0, the radius $k\sin(2\theta)$ increases and then decreases (since k is positive). So we finish the lobe when we get to the second value of θ at which the curve intersects the origin, $\theta = \frac{\pi}{2}$. The final answer is thus $\boxed{0 \le \theta \le \pi/2 \text{ or, equivalently, the closed interval } [0, \pi/2]}$.

c. [4 points] Write, but do **not** evaluate, an integral that gives the area of the top lobe.

Solution: Using the formula for polar area and the values of θ from part ,

$$\text{area} = \int_0^{\pi/2} \frac{1}{2}(f(\theta))^2\, d\theta = \int_0^{\pi/2} \frac{(k\sin(2\theta))^2}{2}\, d\theta.$$

d. [4 points] Imagine that an electron lies within the top lobe of this orbital, but is as far away from the origin as possible. What are the polar coordinates of this point of greatest distance from the origin? Your answer may involve the constant k.

Solution: Maximizing distance from the origin means maximizing $|r|$, so want $\sin(2\theta) = \pm 1$. For θ in the interval $0 \le \theta \le \pi/2$, this implies that $2\theta = \pi/2$ so $\theta = \pi/4$. When $\theta = \pi/4$, we have $r = k\sin(\pi/2) = k$.

Therefore, the polar coordinates for this point are $\boxed{(r, \theta) = \left(k, \frac{\pi}{4}\right)}$.

5. [5 points] Consider the improper integral $\displaystyle\int_1^\infty \frac{2}{3x+5e^x}\,dx$.

Note that for $x > 0$, we have $\displaystyle\frac{2}{3x+5e^x} < \frac{2}{3x}$ and $\displaystyle\frac{2}{3x+5e^x} < 0.4e^{-x}$.

Use this information together with the (Direct) Comparison Test for Integrals to determine whether $\displaystyle\int_1^\infty \frac{2}{3x+5e^x}\,dx$ converges or diverges.

Write the comparison function you use on the blank below and circle your conclusion for the improper integral. Then briefly explain your reasoning.

Answer: Using (direct) comparison of $\displaystyle\frac{2}{3x+5e^x}$ with the function $\underline{\quad 0.4e^{-x} \quad}$,

the improper integral $\displaystyle\int_1^\infty \frac{2}{3x+5e^x}\,dx$ **Converges** **Diverges**

Briefly explain your reasoning.

Solution: Note that the improper integral $\displaystyle\int_1^\infty e^{-x}\,dx$ is one of the "useful integrals for comparison" from the textbook. This integral is known to converge (exponential decay), so the improper integral $\displaystyle\int_1^\infty 0.4e^{-x}\,dx = 0.4\int_1^\infty e^{-x}\,dx$ also converges. Together with the given inequality $\displaystyle\frac{2}{3x+5e^x} < 0.4e^{-x}$ this implies that the improper integral $\displaystyle\int_1^\infty \frac{2}{3x+5e^x}\,dx$ must also converge by the (Direct) Comparison Test for Improper Integrals.

6. [7 points] Consider the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2-n+2}{4n^4-3n^2}$.

Use the **Limit Comparison Test** to determine whether this series converges or diverges.

Circle your answer (either "converges" or "diverges") clearly.

The series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2-n+2}{4n^4-3n^2}$ **Converges** **Diverges**

Give full evidence to support you answer below. Be sure to clearly state your choice of comparison series, show each step of any computation, and carefully justify your conclusions.

Solution: By considering the exponents in the numerator and the denominator of the general term of this series, we decide to compare this series to $\sum_{n=2}^{\infty} \frac{1}{4n^2}$.

$$\lim_{n\to\infty} \frac{(n^2-n+2)/(4n^4-3n^2)}{1/4n^2} = \lim_{n\to\infty} \frac{(4n^2)\cdot(n^2-n+2)}{4n^4-3n^2} = 1.$$

Since this limit exists and is non-zero, we can apply the Limit Comparison Test. So the original series and the series $\sum_{n=2}^{\infty} \frac{1}{4n^2}$ either both converge or both diverge. The series $\sum_{n=2}^{\infty} \frac{1}{4n^2}$ is 1/4 times a p-series ($p = 2$) with the first term ($n = 1$) omitted. Omitting a single term and multiplying by a non-zero constant do not affect the convergence of a series, so since the p-series with $p = 2$ converges, so too will the series $\sum_{n=2}^{\infty} \frac{1}{4n^2}$ converge.

By the Limit Comparison Test, we can therefore conclude that the original series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2-n+2}{4n^4-3n^2}$ must also converge.

7. [12 points] A bouncy ball is launched up 20 feet from the floor and then begins bouncing. Each time the ball bounces up from floor, it bounces up again to a height that is 60% the height of the previous bounce. (For example, when it bounces up from the floor after falling 20 ft, the ball will bounce up to a height of $0.6(20) = 12$ feet.)
Consider the following sequences, defined for $n \geq 1$:

- Let h_n be the height, in feet, to which the ball rises when the ball leaves the ground for the nth time. So $h_1 = 20$ and $h_2 = 12$
- Let f_n be the total distance, in feet, that the ball has traveled (both up and down) when it bounces on the ground for the nth time. For example, $f_1 = 40$ and $f_2 = 40 + 24 = 64$.

a. [2 points] Find the values of h_3 and f_3.

Solution: $h_3 = 0.6(12) = 7.2$ and $f_3 = 64 + 14.4 = 78.4.$

Answer: $h_3 =$ 7.2 and $f_3 =$ 78.4

b. [6 points] Find a closed form expression for h_n and f_n.
("Closed form" here means that your answers should not include sigma notation or ellipses $(\cdots)$. Your answers should also **not** involve recursive formulas.)

Solution: $h_n = 0.6h_{n-1}$ is a recursive relationship that holds between the terms of the sequence h_n for $n > 1$, and this recursive formula means that h_n is a geometric sequence. The (constant) ratio of successive terms is equal to 0.6 and first term is $h_1 = 20$. So we see that $h_n = 20(0.6)^{n-1}$.

Note that the term f_n is twice the sum of the first n terms of the h_n sequence. (Twice because the bouncy ball travels both up and down.) We use the formula for a partial sum of a geometric series (i.e. a finite geometric series) to find

$$\begin{aligned} f_n &= 2(h_1 + h_2 + \ldots + h_n) = 2(20 + \ldots + 20(0.6)^{n-1}) \\ &= \frac{2(20)(1-(0.6)^n)}{1-0.6} = \frac{40(1-(0.6)^n)}{0.4} = 100(1-(0.6)^n). \end{aligned}$$

Answer: $h_n =$ $20 \cdot (0.6)^{n-1}$ and $f_n =$ $\frac{40(1-(0.6)^n)}{0.4} = 100(1-(0.6)^n)$

c. [4 points] Decide whether the given sequence or series converges or diverges.
If it diverges, circle "diverges". If it converges, circle "converges" and write the value to which it converges in the blank.

i. The sequence f_n

Converges to 100 **Diverges**

Solution: The limit of the sequence f_n is

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \frac{40(1-(0.6)^n)}{0.4} = \frac{40}{0.4} = 100.$$

Since this limit exists, the sequence f_n converges, and this computation shows that it converges to 100.

Alternatively, as we saw in part **b**, the sequence f_n is the sequence of partial sums of the geometric series $\sum_{k=1}^{\infty} 2h_k = \sum_{k=1}^{\infty} 40(0.6)^{k-1}$. Since $r = 0.6$ and $|0.6| < 1$, we know that this geometric series converges to $\frac{40}{1-0.6} = 100$. By definition of series convergence, this sum is the limit of the sequence of partial sums f_n, i.e. $\lim_{n\to\infty} f_n = 100$.

ii. The series $\sum_{n=1}^{\infty} h_n$

Converges to 50 **Diverges**

Solution: Next, we consider the series $\sum_{n=1}^{\infty} h_n$, which we know is geometric from part **b**. Since the common ratio between successive terms is 0.6, the series converges, and the formula for the sum of a convergent geometric series gives us

$$\sum_{n=1}^{\infty} h_n = \sum_{n=1}^{\infty} 20 \cdot (0.6)^{n-1} = \frac{20}{1-0.6} = 50,$$

Alternatively, since the sequence f_n is the sequence of partial sums of the series $\sum_{k=1}^{\infty} 2h_k$, we have $\sum_{n=1}^{\infty} h_n = \frac{1}{2} \lim_{n\to\infty} f_n = \frac{100}{2} = 50$.

8. [9 points] For each of parts **a** through **c** below, circle all of the statements that must be true. Circle "NONE OF THESE" if none of the statements must be true.
You must circle at least one choice to receive any credit.
No credit will be awarded for unclear markings. No justification is necessary.

a. [3 points] Suppose $f(x)$ is a continuous and decreasing function on the interval $[0,2]$ with $f(0)=1$ and $f(2)=0$.

Let a be a constant with $0<a<1$. Consider the integral $\displaystyle\int_a^2 \frac{1}{f(x)}\,dx$.

i. This integral is not improper.

ii. This integral converges by direct comparison with the constant function 1.

iii. This integral converges by direct comparison with the function $f(x)$.

iv. This integral converges for some values of a between 0 and 1 but diverges for other values of a between 0 and 1.

v. $\boxed{\text{NONE OF THESE}}$

b. [3 points] Suppose $g(x)$ is a positive and decreasing function that is defined and continuous on the open interval $(5,\infty)$ such that

$\displaystyle\int_{10}^{\infty} g(x)\,dx$ converges and $\displaystyle\int_5^8 g(x)\,dx$ diverges.

i. $\boxed{\text{The series } \displaystyle\sum_{n=20}^{\infty} g(n) \text{ converges.}}$

ii. $\boxed{\text{The series } \displaystyle\sum_{n=12}^{\infty} \frac{1}{g(n)} \text{ diverges.}}$

iii. $\boxed{\text{The sequence } c_n = \displaystyle\int_{15}^{n} g(x)\,dx,\ n\geq 15, \text{ converges.}}$

iv. $\boxed{\text{The integral } \displaystyle\int_5^7 g(x)\,dx \text{ diverges.}}$

v. NONE OF THESE

c. [3 points] Consider the sequence $a_n = \dfrac{1}{\ln(n)}$, $n\geq 2$.

Note: Due to a typo on the original exam (corrected here), all students received full credit for part **c**.

i. $\boxed{\displaystyle\lim_{n\to\infty} a_n = 0.}$

ii. The series $\displaystyle\sum_{n=2}^{\infty} a_n$ converges.

iii. $\boxed{\text{The series } \displaystyle\sum_{n=2}^{\infty} a_n \text{ diverges.}}$

iv. $\boxed{\text{The series } \displaystyle\sum_{n=2}^{\infty} (-1)^n a_n \text{ converges.}}$

v. NONE OF THESE

9. [12 points] For each of parts **a** through **c** below, determine the radius of convergence of the power series. Show your work carefully.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty} \frac{e}{n!}(x-1)^n$

Solution: We apply the Ratio Test.

$$\lim_{n\to\infty} \frac{|(e(x-1)^{n+1})/(n+1)!|}{|(e(x-1)^n)/n!|} = \lim_{n\to\infty} \frac{e}{e} \cdot \frac{n!}{(n+1)!} \cdot |x-1| = \lim_{n\to\infty} \frac{1}{n+1}|x-1| = 0.$$

This limit is always less than one, so, by the Ratio Test, this power series will converge for every value of x. Hence the radius of convergence is ∞.

Answer: radius of convergence = ∞

b. [3 points] $5(x+\pi) + 5\cdot 4(x+\pi)^2 + 5\cdot 9(x+\pi)^3 + 5\cdot 16(x+\pi)^4 + \cdots$

Solution: We apply the Ratio Test.

$$\lim_{n\to\infty} \frac{|5(n+1)^2(x+\pi)^{n+1}|}{|5n^2(x+\pi)^n|} = \lim_{n\to\infty} \frac{n+1}{n}|x+\pi| = |x+\pi|.$$

The Ratio Test guarantees convergence when this limit is less than one (and divergence when the limit is greater than one). Now $|x+\pi| < 1$ means x is within 1 unit of π (or $-\pi - 1 < x < -\pi + 1$), so the radius of convergence is 1.

Answer: radius of convergence = 1

c. [3 points] $\displaystyle\sum_{n=0}^{\infty} \frac{\pi}{8^n}(x+2)^{3n}$

Solution: Note that this series is geometric with first term π and ratio of successive terms $\frac{(x+2)^3}{8}$, so the series converges if and only if $\left|\frac{(x+2)^3}{8}\right| < 1$.
Alternatively, we can apply the Ratio Test:

$$\lim_{n\to\infty} \frac{|\pi(x+2)^{3(n+1)}/8^{n+1}|}{|\pi(x+2)^{3n}/8^n|} = \lim_{n\to\infty} \frac{8^n}{8^{n+1}}|x+2|^3 = \frac{|x+2|^3}{8}$$

The Ratio Test guarantees convergence when this limit is less than one (and divergence when it is greater than one).

Using either approach, we see that the power series converges when $\dfrac{|x+2|^3}{8} < 1$, i.e. when $|x+2| < 2$.

Answer: radius of convergence = 2

d. [3 points] Consider the power series $\sum_{j=0}^{\infty} C_j(x-5)^j$, where each C_j is a constant. Suppose this power series

- converges when $x = 2$ and
- diverges when $x = 12$.

Based on this information, which of the following values **could** be equal to the radius of convergence of the power series? Circle all possibilities from the list below.

0	1	2	$\boxed{3}$	$\boxed{4}$
$\boxed{5}$	$\boxed{6}$	$\boxed{7}$	8	9
10	11	12	NONE OF THESE	

10. [9 points] The sequence $\{\gamma_n\}$ is defined according to the formula

$$\gamma_n = -\ln(n) + \sum_{k=1}^{n} \frac{1}{k}.$$

(You may recall this sequence from team homework 5.) This sequence converges to a positive number γ (which happens to be $\gamma \approx 0.5772156649$).

a. [2 points] Does the sequence $\{\gamma_n^2\}$ converge or diverge? If this sequence converges, compute the value to which this sequence converges, either in terms of the constant γ or with five decimal places of accuracy.

Solution: Yes, this sequence converges.

$$\lim_{n\to\infty} \gamma_n^2 = \left(\lim_{n\to\infty} \gamma_n\right)^2 = \gamma^2 \approx 0.33318.$$

b. [3 points] Does the series $\sum_{n=1}^{\infty} \gamma_n$ converge or diverge? Briefly explain your answer, and if this series converges, compute the value to which the series converges either in terms of the constant γ or with five decimal places of accuracy.

Solution: This series diverges by the n^{th} term test for divergence. The terms of this series do not approach 0; instead the terms approach γ, i.e.

$$\lim_{n\to\infty} \gamma_n = \gamma \approx 0.5772156649 \neq 0.$$

c. [4 points] Let $h_n = \sum_{k=1}^{n} \frac{1}{k}$. Find the value of $\lim_{n\to\infty} \frac{e^{h_n}}{n}$.

You may give your answer either in terms of the constant γ or with five decimal places of accuracy.

Hint: First consider $\lim_{n\to\infty} \ln\left(\frac{e^{h_n}}{n}\right)$.

Solution: Following the hint, we first compute

$$\ln\left(\frac{e^{h_n}}{n}\right) = \ln\left(\frac{e^{\sum_{k=1}^{n} 1/k}}{n}\right) = \ln(e^{\sum_{k=1}^{n} 1/k}) - \ln(n) = -\ln(n) + \sum_{k=1}^{n} \frac{1}{k} = \gamma_n.$$

Next, we take the limit of this expression.

$$\lim_{n\to\infty} \ln\left(\frac{e^{h_n}}{n}\right) = \lim_{n\to\infty} \gamma_n = \gamma \approx 0.5772156649.$$

Finally, the limit we are looking for is found by exponentiating this result (in order to "undo" the natural log from before).

$$\lim_{n\to\infty} \frac{e^{h_n}}{n} = e^{\gamma} \approx e^{0.5772156649} \approx 1.78107.$$