On my honor, as a student,
I have neither given nor received
unauthorized aid on this academic work. **Initials:** ______

Do not write in this area

Math 116 — Final Exam — December 14, 2017

Your Initials Only: ______ Your U-M ID # (not uniqname): ______________

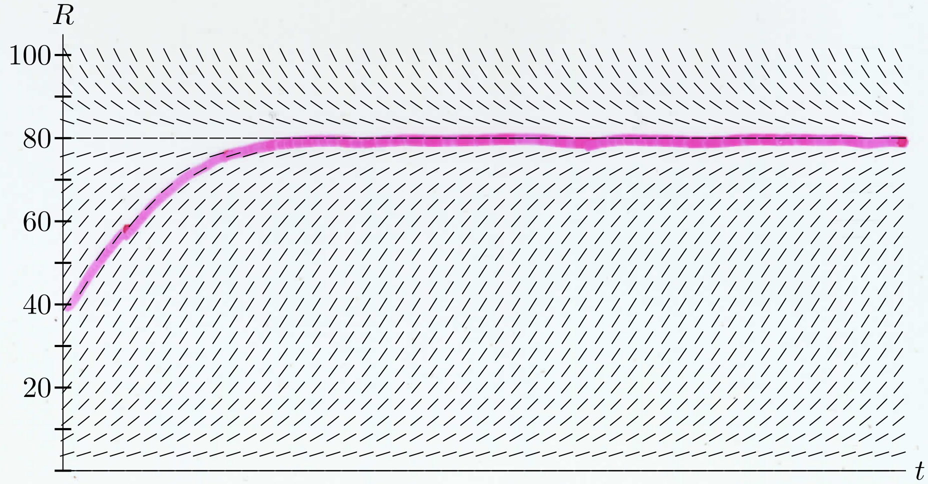
Instructor Name: UNOFFICIAL SOLUTIONS Section #: ______

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 14 pages including this cover. The last page provides some potentially useful formulas. You may separate the formula page from the exam, but please do turn it in along with the exam. Do not separate the other pages. If pages do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
4. There are 12 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form.* Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	6	
2	8	
3	11	
4	4	
5	5	
6	8	

Problem	Points	Score
7	9	
8	11	
9	6	
10	12	
11	14	
12	6	
Total	100	

1. [6 points] Suppose that a population of foxes is introduced into a small ecosystem in order to help control the rabbit population. Let R be the population of rabbits t days after the introduction of the foxes. The slope field for a differential equation modeling the rabbit population over time is shown below.



a. [2 points] Find all nonzero equilibrium solutions to the differential equation corresponding to this slope field. Write your answer as one or more equations of the form $R = f(t)$. If there are no nonzero equilibrium solutions, explain how you can tell.

$\boxed{R=0 \quad R=80}$

b. [2 points] On the slope field above, sketch the **particular solution** that describes what happens to an initial rabbit population of 40 rabbits.

c. [2 points] Let $R = g(t)$ be the particular solution that you sketched in part (b). Compute $\lim_{t\to\infty} g(t)$.

Answer: $\lim_{t\to\infty} g(t) =$ $\boxed{80}$

Complete the sentence below to give a practical interpretation in the context of this problem of the limit you found. Be sure to fill in the blank.

Interpretation: *If there were* 40 *rabbits at the time when the foxes were introduced to the ecosystem, then*

the population of rabbits will increase, and approach 80 in the long run.

2. [8 points] For this problem, consider the family of polar curves described for each positive integer $n \geq 1$ by

$$r = \frac{\cos(2n\theta)}{n}$$

for $0 \leq \theta \leq 2\pi$.

a. [2 points] Consider the polar curve described by $r = \cos(2\theta)$ for $0 \leq \theta \leq 2\pi$. (Note that this is the case of $n = 1$.) Find all values of θ between 0 and 2π for which the curve $r = \cos(2\theta)$ passes through the origin.

At origin $\iff r=0 \iff \cos(2\theta)=0$

$\iff 2\theta = m\pi + \frac{\pi}{2}$ for some integer m

$\iff \theta = m\frac{\pi}{2} + \frac{\pi}{4}$ for some integer m

Answer: $\theta = \boxed{\frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}}$

b. [3 points] For $n \geq 1$, find all x-intercepts of the polar curve $r = \dfrac{\cos(2n\theta)}{n}$. Your answer(s) may involve n.

x-intercept $\iff 0 = y = r\sin\theta \iff r=0$ or $\sin\theta = 0$.
r can be 0, as when $\theta = \pi/4n$. And $\sin\theta = 0$ when $\theta = m\pi$ for some integer m. In that case, $r = \dfrac{\cos(2nm\pi)}{n} = \dfrac{1}{n}$ and $\cos\theta = \begin{cases} 1 & \text{if } m \text{ even} \\ -1 & \text{if } m \text{ odd.} \end{cases}$

So $x = \pm\frac{1}{n}$.

Answer: $x = \boxed{-\frac{1}{n},\ 0,\ \frac{1}{n}}$

c. [3 points] For $n \geq 1$, let A_n be the arclength of the polar curve $r = \dfrac{\cos(2n\theta)}{n}$ for $0 \leq \theta \leq 2\pi$. Write, but do not evaluate, an expression involving one or more integrals that gives the value of A_n.

$$A_n = \int_0^{2\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

where $f(\theta) = \frac{1}{n}\cos(2n\theta)$

$f'(\theta) = -2\sin(2n\theta)$

Answer: $A_n = \boxed{\int_0^{2\pi} \sqrt{\frac{1}{n^2}\cos^2(2n\theta) + 4\sin^2(2n\theta)}\, d\theta}$

3. [11 points] Show your work, but no explanation is necessary. For parts **a**, **c**, and **d**, be sure to pay close attention to whether the question is asking you for a median or a mean.

a. [3 points] Compute the **median** value of a quantity that has cumulative distribution function given by

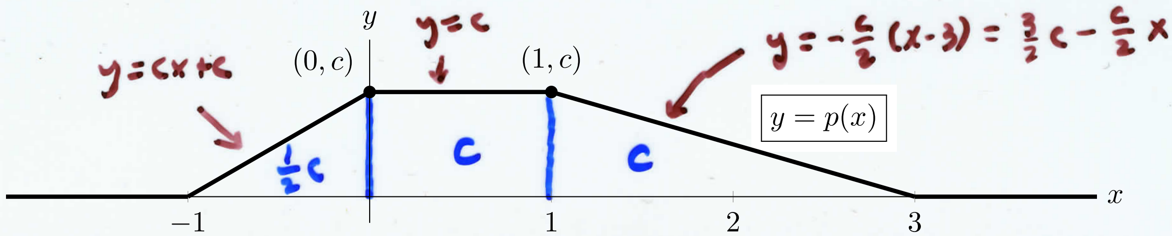
CDF: $$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-(x/r)^k} & \text{if } x \geq 0 \end{cases}$$

Here r and k are constants, and your answer may involve one or both of these constants.

$$\frac{1}{2} = F(x) = 1 - e^{-(x/r)^k} \Rightarrow e^{-(x/r)^k} = \frac{1}{2} \Rightarrow -\left(\frac{x}{r}\right)^k = \ln\frac{1}{2} = -\ln 2$$

Answer: median = $\boxed{r(\ln 2)^{1/k}}$

Use the probability density function $p(x)$ shown in the graph below for parts **b-d**.



b. [2 points] Use the fact that the graph above shows a probability density function to find the value of the constant c.

$$1 = \text{area under pdf} = \frac{1}{2}c + c + c = \frac{5}{2}c$$

Answer: $c =$ $\boxed{2/5}$

c. [3 points] Compute the **mean** of the quantity with probability density function shown in the graph above.

$$\text{mean} = \int_{-\infty}^{\infty} x p(x)\,dx = \int_{-1}^{0} x(cx+c) + \int_0^1 x(c) + \int_1^3 x\left(\frac{3}{2}c - \frac{c}{2}x\right)$$

$$= c\left[\int_{-1}^{0} x^2 + x + \int_0^1 x + \int_1^3 \frac{3}{2}x - \frac{1}{2}x^2\right] = \frac{2}{5}\left[-\frac{1}{6} + \frac{1}{2} + \frac{5}{3}\right]$$

Answer: mean = $\boxed{4/5}$

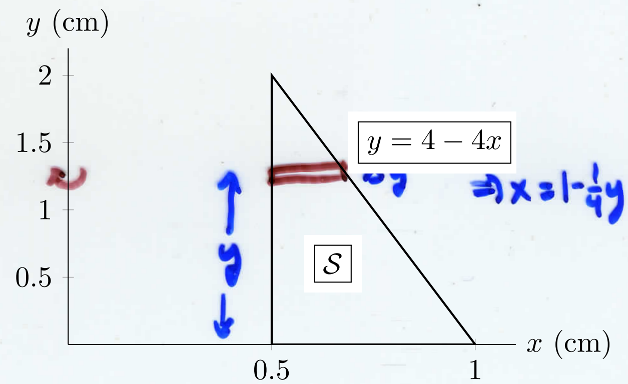
d. [3 points] Compute the **median** of the quantity with probability density function shown in the graph above. Guess median is between 0 and 1.

Then $$\frac{1}{2} = \frac{1}{2}c + \int_0^m c = \frac{1}{2}c + mc \Rightarrow m = \frac{\frac{1}{2} - \frac{1}{2}c}{c} = \frac{\frac{1}{2} - \frac{1}{5}}{2/5}$$

Answer: median = $\boxed{3/4}$

4. [4 points]

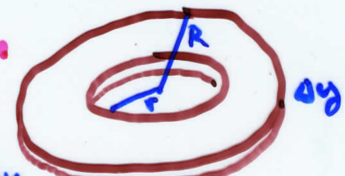
Let $\mathcal{S}$ be the region bounded by the x-axis, the line $x = 0.5$, and the line $y = 4 - 4x$. This region is shown to the right. The units on both the x- and the y-axis are centimeters. A solid is obtained by rotating the region $\mathcal{S}$ about the y-axis. The mass density of the resulting solid at each point y centimeters above the x-axis is $16y$ grams per cubic centimeter.



Write, but do **not** evaluate, an expression involving one or more integrals that gives the mass, in grams, of the resulting solid.

inner radius of slice $= r = 0.5$ cm

outer " " " $= R = 1 - \frac{1}{4}y$ cm



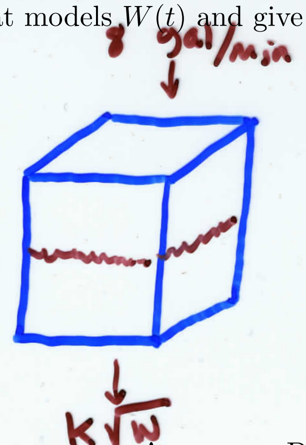
Volume " " $= \pi R^2 \Delta y - \pi r^2 \Delta y$

$= \pi [R^2 - r^2] \Delta y = \pi [(1 - \frac{1}{4}y)^2 - (.5)^2] \Delta y$ cm^3

mass = Volume · density $= \pi((1 - \frac{1}{4}y)^2 - (\frac{1}{2})^2) \Delta y (16y)$ g

Answer: Mass = $\boxed{16\pi \int_0^2 \left((1 - \frac{1}{4}y)^2 - (\frac{1}{2})^2\right) y \, dy}$ g

5. [5 points] Prior to the start of an indoor winter carnival, the water tank for a dunking booth is being filled from a water hose at a rate of 8 gallons per minute. Unfortunately, once the tank has 50 gallons of water in it, the tank begins leaking water at a rate (in gallons per minute) that is proportional to the square root of the volume of water in the tank (in gallons) with constant of proportionality $k > 0$. Let $W = W(t)$ be the volume, in gallons, of water that is in the tank t minutes after the tank begins to leak. Write a differential equation that models $W(t)$ and give an appropriate initial condition.



Answer: Differential Equation: $\frac{dW}{dt} = 8 - k\sqrt{W}$

Initial Condition: $W(0) = 50$

6. [8 points]

Values of a function f and some of its derivatives are given in the table on the right. Use this information to answer the questions that follow.

x	0	π
$f(x)$	-6	2π
$f'(x)$	6	2
$f''(x)$	1	-3
$f'''(x)$	-1	0
$f''''(x)$	5	$-9/2$

a. [4 points] Find a formula for the Taylor polynomial of degree 4 for f about $x = \pi$.

$$P_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n$$

$$= f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 + \frac{f'''(\pi)}{3!}(x-\pi)^3 + \frac{f^{IV}(\pi)}{4!}(x-\pi)^4$$

$$= 2\pi + 2(x-\pi) + \frac{-3}{2}(x-\pi)^2 + \frac{0}{6}(x-\pi)^3 + \frac{-9/2}{24}(x-\pi)^4$$

$$= \boxed{2\pi + 2(x-\pi) - \frac{3}{2}(x-\pi)^2 - \frac{3}{16}(x-\pi)^4}$$

b. [4 points] Find the first three nonzero terms of the Taylor series for $\displaystyle\int_0^x f(t^2)\,dt$ about $x = 0$.

Near 0, $f(x) \approx f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$

$$= -6 + 6x + \frac{1}{2}x^2$$

So $\displaystyle\int_0^x f(t^2)\,dt \approx \int_0^x -6 + 6t^2 + \frac{1}{2}t^4\,dt$

$$= -6t + 2t^3 + \frac{1}{10}t^5 \Big|_0^x$$

$$= \boxed{-6x + 2x^3 - \frac{1}{10}x^5}$$

7. [9 points] The "Witch of Maria Agnesi" is the family of curves described by the parametric equations

$$x = 2at \qquad \text{and} \qquad y = \frac{2a}{1+t^2} = 2a(1+t^2)^{-1}$$

for all t, where a is a positive constant.

a. [4 points] Consider a "Witch curve" as defined above.

i. Find a formula for $\frac{dy}{dx}$ in terms of t and/or a.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\cancel{2a}(1+t^2)^{-2}(2t)}{\cancel{2a}}$$

Answer: $\frac{dy}{dx}$ $\boxed{\frac{-2t}{(1+t^2)^2}}$

ii. Find a formula for the tangent line to the "Witch curve" at the point where $t = 1$. Your answer might involve the constant a but should **not** involve t.

at $t=1$:

$x = 2a(1) = 2a$

$y = \frac{2a}{1+(1)^2} = a$

$\frac{dy}{dx} = \frac{-2(1)}{(1+1^2)^2} = -\frac{1}{2}$

Answer: $y =$ $\boxed{a - \frac{1}{2}(x-2a)}$

b. [5 points] The total area in the first quadrant that is bounded between a "Witch curve" and the x-axis is represented by the improper integral $\displaystyle\int_0^\infty \frac{8a^3}{x^2+4a^2}\,dx$.

Determine whether this improper integral converges or diverges.

- If the integral converges, circle "converges", find its <u>exact value</u> (i.e. no decimal approximations), and write the exact value on the answer blank provided. (The exact value may involve the constant a.)
- If the integral diverges, circle "diverges" and carefully justify your answer.

In either case, **you must show all your work carefully using correct notation**. Any direct evaluation of integrals must be done **without using a calculator.**

Converges to $\boxed{2\pi a^2}$ **Diverges**

Hint: Note that $\frac{d}{dx}\left(4a^2 \arctan\left(\frac{x}{2a}\right)\right) = \frac{8a^3}{x^2+4a^2}$.

$$\int_0^\infty \frac{8a^3}{x^2+4a^2}\,dx = \lim_{b\to\infty}\left[4a^2 \arctan\left(\frac{x}{2a}\right)\right]_0^b$$

$$= 4a^2 \lim_{b\to\infty} \arctan\left(\frac{b}{2a}\right) - \cancel{\arctan(0)}^{\,0} = 4a^2\left(\frac{\pi}{2}\right)$$

8. [11 points] Imagine that a one pound ball is attached to a spring. This ball is allowed to move forward and backward on a table, but not up and down (or side to side). When the spring is not stretched at all, we say that the ball is at its *starting position*. Let x be the displacement of the ball from its starting position in the forward/backward direction. (The value of x is positive if the ball has moved forward from its starting position and negative if the ball has moved backward from its starting position.)

a. [4 points] Let $F(x)$ be the magnitude of the force, measured in pounds, that the spring exerts on the ball when the ball has been pulled x feet from its starting position. Suppose $F(x) = 5x$.

i. Which of the following best estimates the work, in foot-pounds, needed to move the ball a very small distance Δx feet forward from a position x? *Circle* ONE *choice.*

I. 5 II. $5x$ III. $2.5x^2$ IV. $5\Delta x$ V. $5x\Delta x$ VI. $2.5x^2\Delta x$

ii. Use your answer to part i. to write an expression involving one or more integrals that gives the total work needed to move the ball from its starting position forward a distance of one half of one foot (i.e. 6 inches). Then compute the value of your integral (either by hand or using your calculator). Include units on your answer.

$\int_0^{.5} 5x\,dx = \frac{5}{2}x^2\Big|_0^{.5} = \frac{5}{2}\left(\frac{1}{2}\right)^2$

Answer: Integral Expression: $\boxed{\int_0^{.5} 5x\,dx}$

Numerical Answer (with units): $\boxed{5/8 \text{ Foot pounds}}$

b. [4 points] After stretching the spring as described above, you release it from a starting position of $x = 1/2$. The ball oscillates backwards and forwards (in the x-direction), and its position $x = x(t)$ satisfies the differential equation $x'' + 5x = 0$. Note that $x'' = \frac{d^2x}{dt^2}$. For what values of A, B, and k will the function

$$x(t) = A\sin(kt) + B\cos(kt)$$

be a solution to the differential equation $x'' + 5x = 0$ with the initial conditions $x(0) = 1/2$ and $x'(0) = 0$?

Since $x'' = -k^2 x$, $k^2 = 5$

$x'(t) = kA\cos(kt) - kB\sin(kt)$ $\frac{1}{2} = x(0) = B$

$x''(t) = -k^2 A\sin(kt) - k^2 B\cos(kt)$ $0 = x'(0) = kA$

Answer: $A = \boxed{0}$ and $B = \boxed{1/2}$ and $k = \boxed{\pm\sqrt{5}}$

c. [3 points] Using the particular solution that you found in part **b**, find the first time $t > 0$ when the ball reaches the position $x = 0$.

$x(t) = \frac{1}{2}\cos(\sqrt{5}t) = 0 \Rightarrow \sqrt{5}t = \frac{\pi}{2} + m\pi$

Answer: $t = \boxed{\pi/2\sqrt{5}}$

9. [6 points] Suppose that a psychology experiment is designed so that every participant receives a score that is some (positive or negative) real number. The score that a participant receives is called his or her "experimental score". The experiment is calibrated so that the probability density function of the distribution of experimental scores is standard normal, call it $g(x)$.

a. [3 points] Use a complete sentence to give a practical interpretation of the integral expression

$$\int_{-4}^{3} g(x)\,dx$$

in the context of this problem.

$\int_{-4}^{3} g(x)\,dx$ is the probability that a score is (or the proportion of scores that are) between -4 and 3.

b. [3 points] Note that $g(1) = \frac{1}{\sqrt{2e\pi}} \approx 0.24$. Which of the following is the best practical interpretation of the mathematical statement $g(1) \approx 0.24$? Circle the ONE best option.

i. The fraction of the population having experimental score equal to 1 is approximately 24 percent.

ii. The fraction of the population having experimental score equal to 0.24 is approximately 1 percent.

iii. The fraction of the population having experimental score between 0.9 and 1.1 is approximately 4.8 percent.

iv. The fraction of the population having experimental score between 0.23 and 0.25 is approximately 20 percent.

v. The fraction of the population having experimental score above 1 is approximately 76 percent.

vi. The fraction of the population having experimental score above 0.24 is approximately 1 percent.

$P(0.9 < \text{score} < 1.1) = \int_{0.9}^{1.1} g(x)\,dx$. The MID(1) approximation of that integral is $(0.2)g(1) \approx 0.048$.

10. [12 points]

a. [6 points] Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n \cdot 3^n}.$$

Show every step of any calculations and fully justify your answer with careful reasoning. Write your final answers on the answer blanks provided.

Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2} x^{2(n+1)}}{(n+1)\, 3^{n+1}} \cdot \frac{n \cdot 3^n}{(-1)^{n+1} x^{2n}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(-1)^{n+2}}{(-1)^{n+1}} \cdot \frac{x^{2n+2}}{x^{2n}} \cdot \frac{n}{n+1} \cdot \frac{3^n}{3^{n+1}}\right| = \left|\frac{-x^2}{3}\right| \lim_{n\to\infty} \frac{n}{n+1} \; 1$$

$= x^2/3$, which is <1 when $|x| < \sqrt{3}$.

$x = \pm\sqrt{3} \Rightarrow$ sum $= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ which converges by alternating series test

AST:
alternating ✓
|terms| decreasing ✓
terms → 0 ✓

Answer: Radius of Convergence: $\boxed{\sqrt{3}}$

Interval of Convergence: $\boxed{[-\sqrt{3}, \sqrt{3}]}$

b. [3 points] The Maclaurin series for a function $f(x)$ is the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n \cdot 3^n}$.

The function $f(x)$ is closely related to one of the functions that appears on the formula sheet on the last page of this exam. Find a formula for $f(x)$ in closed form (i.e. without sigma notation or ellipses $(\cdots)$).

$$\sum_{1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n \cdot 3^n} = \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{x^2}{3}\right)^n = \ln\left(1 + \frac{x^2}{3}\right)$$

Answer: $f(x) = \boxed{\ln(1 + x^2/3)}$

c. [3 points] Suppose the Taylor series about $x = 0$ for a function $g(x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} x^{3n}$.

Let $h(x) = g(\frac{x}{2})$. Find $h^{(15)}(0)$.

$$h(x) = \sum \frac{(-1)^n}{n^2} \left(\frac{x}{2}\right)^{3n} = \sum \frac{(-1)^n}{n^2 (2^{3n})} x^{3n}.$$

$\frac{h^{(15)}(0)}{15!}$ = coefficient of x^{15} $= \frac{(-1)^5}{5^2 (2^{15})}$

Answer: $h^{(15)}(0) = \boxed{-\frac{15!}{5^2 2^{15}}}$

11. [14 points] An auto manufacturer is testing the braking capability of one of its hybrid-electric vehicles. At regular time intervals during the experiment, the auto engineers measure the speed and the position of the car along the test track.
Let t be the number of seconds after the car begins braking.

Let $v(t)$ be the car's speed at time t, in meters per second, and let $p(t) = \int_0^t v(s)\,ds$.

The auto engineers are most interested in the time period $0 \le t \le 40$, when the car's acceleration is always negative but increasing.
The velocity measurements taken during this time period are given in the table below.

t (seconds)	0	10	20	30	40
$v(t)$ (m/s)	111	60	25	5	0

a. [3 points] Consider the four approximations of the definite integral $\int_0^{40} v(t)\,dt$ given by RIGHT(4), LEFT(4), TRAP(4), and MID(4). Rank these five quantities in order from least to greatest by filling in the blanks below with the options I–V.

v' negative but increasing $\Rightarrow$ v dec, conc up

I. $\int_0^{40} v(t)\,dt$ II. RIGHT(4) III. LEFT(4)

IV. TRAP(4) V. MID(4)

II < V < I < IV < III

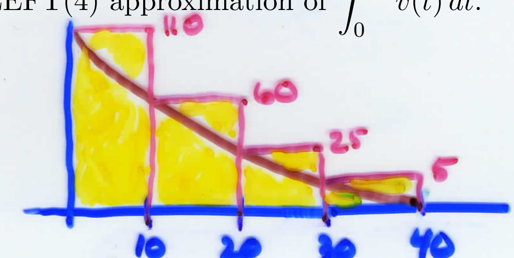
RIGHT < $\int$ since dec; MID < $\int$ since V conc up; $\int$; TRAP > $\int$ since V conc up; LEFT > $\int$ since dec

b. [3 points] Write out all the terms of the LEFT(4) approximation of $\int_0^{40} v(t)\,dt$.

$\Delta t = \frac{40-0}{4} = 10$

LEFT(4) $= \Delta t\,[v_0 + v_1 + v_2 + v_3]$

$\boxed{= 10\,[111 + 60 + 25 + 5]}$



c. [4 points] Let $h(x)$ be the gasoline fuel efficiency of the test vehicle, in liters per hectokilometer (i.e. liters per 100 km) when the car is traveling at a speed of x m/s.

i. Suppose a formula for h is given by $h(x) = 2.3 + 0.097x$.
Compute the value of $\int_0^{40} h'(v(t)) \cdot v'(t)\,dt$.

Let $w = v(t)$
$dw = v'(t)\,dt$
$t = 0 \Rightarrow w = 111$
$t = 40 \Rightarrow w = 0$

$= \int_{111}^{0} h'(w)\,dw = h(w)\Big|_{111}^{0}$

$= h(0) - h(111) = 2.3 - (2.3 + 10.767)$

Answer: $\int_0^{40} h'(v(t)) \cdot v'(t)\,dt =$ $\boxed{-10.767}$

This is a continuation of the problem from the previous page.

ii. Let

$$K = \int_0^{40} h'(v(t)) \cdot v'(t)\, dt$$

(Note that K is the value you computed in part c(i).)
Circle the phrase below that best completes the practical interpretation of K that begins "*During the last 40 seconds of the experiment...*"

I. the vehicle consumes $|K|$ liters of fuel per hectokilometer.

II. the rate of change of the vehicle's fuel efficiency is K liters per hectokilometer per second.

III. the vehicle consumes $|K|$ liters of fuel.

IV. the total change in the rate of change of fuel in the vehicle's gas tank is $1/K$ liters per second.

V. the total change in the vehicle's fuel efficiency is K liters per hectokilometers.

d. [4 points] The energy density of the car's battery is a function of time, $E(t)$, which can be multiplied by the car's position function $p(t)$ in order to compute the battery's charge. Suppose that $E(0) = 1$, $E(40) = 0.89$, $E'(0) = -0.0028$, and $E'(40) = -0.025$. Use your answer to part **b** above to estimate the value of

$$\int_0^{40} \big(v(t)E(t) + p(t)E'(t)\big)\, dt.$$

Hint: What is $p'(t)$?

$p(t) = \int_0^t v(s)ds$

so $p'(t) = v(t)$

we recognize the product rule:

$\int_0^{40} (v(t)E(t) + p(t)E'(t))dt$

$= \int_0^{40} (p'(t)E(t) + p(t)E'(t))dt = \int_0^{40} \frac{d}{dt}[p(t)E(t)]dt$

$= p(t)E(t)\Big|_0^{40} = p(40)E(40) - p(0)E(0)$. $p(0) = 0$ and

$E(40) = .89$, so that's $.89p(40) - 0 = .89\int_0^{40} v(s)ds$

$\approx .89$ (estimate from part b) $= .89(2010) = \boxed{1788.9}$

12. [6 points] For each of the questions below, circle all of the available correct answers. Circle "NONE OF THESE" if none of the available choices are correct. You must circle at least one choice to receive any credit. No credit will be awarded for unclear markings. No justification is necessary.

a. [2 points] Which of the following converge to the number $\int_0^{\pi/2} e^{-x^2}\,dx$?

i. The sequence LEFT(n) of approximations of $\int_0^{\pi/2} e^{-x^2}\,dx$, for $n \geq 1$.

ii. The sequence $\int_{1/n}^{\pi/2} e^{-x^2}\,dx$ for $n \geq 1$

iii. The series $\sum_{n=0}^{\infty} \frac{(-1)^n(\pi^{2n+1})}{(2n+1)(2^{2n+1})(n!)}$

iv. NONE OF THESE

Let $F(x) = \sum_0^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}$. Then $F'(x) = \sum \frac{(-1)^n x^{2n}}{n!} = \sum \frac{(-x^2)^n}{n!} = e^{-x^2}$. So $\int_0^{\pi/2} e^{-x^2} = F(\frac{\pi}{2}) - F(0)$.

b. [2 points] Consider the sequence $a_n = \frac{1}{\ln(n)}$, $n \geq 2$. Which of the following statements are true?

i. $\lim_{n\to\infty} a_n = 0$.

$\ln(n) \to \infty$ so $\frac{1}{\ln(n)} \to 0$

ii. The series $\sum_{n=2}^{\infty} a_n$ converges.

$\ln(n) < n \Rightarrow \frac{1}{\ln(n)} > \frac{1}{n}$. Since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{\ln(n)}$ diverges by comparison.

iii. The series $\sum_{n=2}^{\infty} a_n$ diverges.

iv. The series $\sum_{n=2}^{\infty} (-1)^n a_n$ converges.

conv by alt. series test

v. NONE OF THESE

c. [2 points] Which of the following series are conditionally convergent?

i. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conv by AST, but $\sum \frac{1}{n}$ div by integral test

ii. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ absolutely convergent by p-test (or integral test)

iii. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$ abs convergent by ratio test (geometric series $\to \frac{-1/3}{1+1/3} = -\frac{1}{4}$)

iv. NONE OF THESE

"Known" Maclaurin Series

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\,x^2 + \frac{p(p-1)(p-2)}{3!}\,x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Normal Distributions

The density function of a normal distribution with mean μ and standard deviation $\sigma > 0$ is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

The standard normal distribution is the normal distribution with $\mu = 0$ and $\sigma = 1$.