On my honor, as a student,
I have neither given nor received
unauthorized aid on this academic work. **Initials:** ______

Do not write in this area

Math 116 — First Midterm — October 8, 2018

Your Initials Only: ______ Your U-M ID # (not uniqname): ____________

Instructor Name: UNOFFICIAL SOLUTIONS Section #: ________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. The last page provides some potentially useful formulas. You may separate the formula page from the exam, but please do turn it in along with the exam. Do not separate the other pages. If pages do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
4. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is <u>not</u> permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form.* Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	12	
3	8	
4	12	
5	7	

Problem	Points	Score
6	12	
7	12	
8	12	
9	13	
Total	100	

1. [12 points] The table below gives several values of a differentiable function f such that f' is also differentiable and f'' is continuous.

x	-3	-2	-1	0	1	2	3
$f(x)$	14	20	4	11	24	5	8
$f'(x)$	3	-4	-6	2	5	-3	4

For each of the following, calculate the exact numerical value of the integral. If it is not possible to do so based on the information provided, write "NOT POSSIBLE" and clearly indicate why it is not possible. Show your work.
Note that no variables or function names (such as f, f', or f'') should appear in your answers.

a. [3 points] $\int_0^1 f'(2x)\,dx$

let $w = 2x$
$dw = 2dx$
$x = 0 \Rightarrow w = 0$
$x = 1 \Rightarrow w = 2$

$= \int_0^2 f'(w)\cdot\frac{dw}{2}$

$= \frac{1}{2}\left[f(w)\right]_0^2 = \frac{1}{2}\left[f(2)-f(0)\right] = \frac{1}{2}\left[5-11\right]$

Answer: -3

b. [3 points] $\int_2^3 sf''(s)\,ds$ $= \int_2^3 uv' = uv\Big|_2^3 - \int_2^3 u'v = sf'(s)\Big|_2^3 - \int_2^3 (1)(f'(s))\,ds$

Parts:
Let $u = s$ $\quad v' = f''(s)$
$u' = 1$ $\quad v = f'(s)$

$= sf'(s) - f(s)\Big|_2^3 = \left[3f'(3) - f(3)\right] - \left[2f'(2) - f(2)\right]$

$= \left[3(4) - 8\right] - \left[2(-3) - 5\right] = \left[12-8\right] - \left[-6-5\right] = 4 + 11$

Answer: 15

c. [3 points] $\int_{-2}^{-1} q\cdot\left[\frac{d}{dq}\left(f'(q)e^{f(q)}\right)\right]dq$ $= \int_{-2}^{-1} uv' = uv\Big|_{-2}^{-1} - \int_{-2}^{-1} u'v.$

①, ②

Parts:
Let $u = q$ $\quad v' = \frac{d}{dq}\left(f'(q)e^{f(q)}\right)$
$u' = 1$ $\quad v = f'(q)e^{f(q)}$

① $uv\Big|_{-2}^{-1} = qf'(q)e^{f(q)}\Big|_{-2}^{-1} = (-1)f'(-1)e^{f(-1)} - (-2)f'(-2)e^{f(-2)}$
$= (-1)(-6)e^4 - (-2)(-4)e^{20} = 6e^4 - 8e^{20}$

② $\int_{-2}^{-1} u'v = \int_{-2}^{-1} (1)\,f'(q)e^{f(q)}\,dq$

let $w = f(q)$
$dw = f'(q)\,dq$
$q = -2 \Rightarrow w = f(-2) = 20$
$q = -1 \Rightarrow w = f(-1) = 4$

$= \int_{20}^{4} e^w\,dw = e^w\Big|_{20}^{4} = e^4 - e^{20}$

So we have ① − ② $= (6e^4 - 8e^{20}) - (e^4 - e^{20})$

Answer: $5e^4 - 7e^{20}$

d. [3 points] $\int_{-1}^{1} f'(y)\cdot f''(f(y))\,dy$

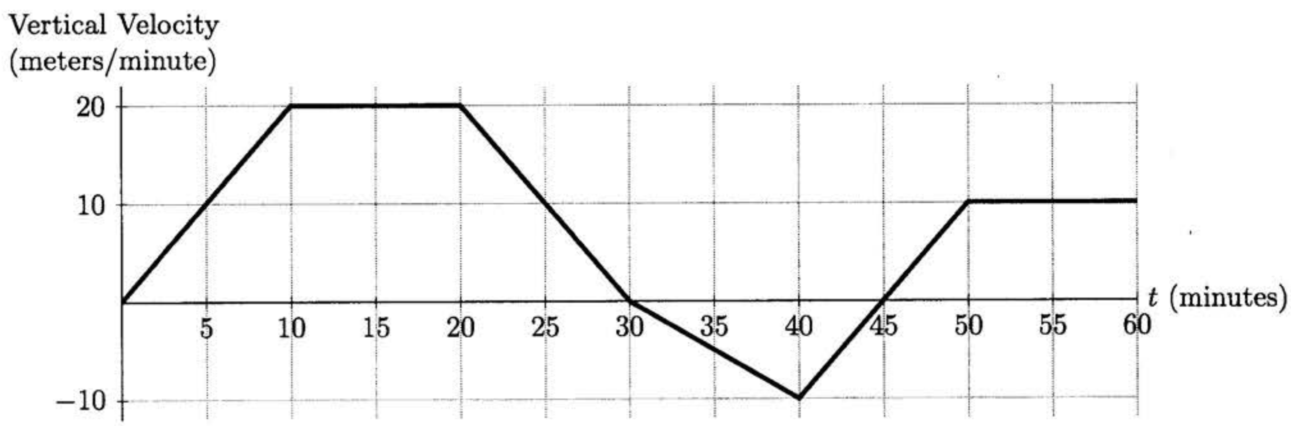
Let $w = f(y)$
$dw = f'(y)\,dy$
$y = -1 \Rightarrow w = f(-1) = 4$
$y = 1 \Rightarrow w = f(1) = 24$

$= \int_4^{24} f''(w)\,dw = f'(w)\Big|_4^{24} = f'(24) - f'(4).$

But we don't know $f'(24)$ and $f'(4)$, so we can't go any further.

Answer: NOT POSSIBLE

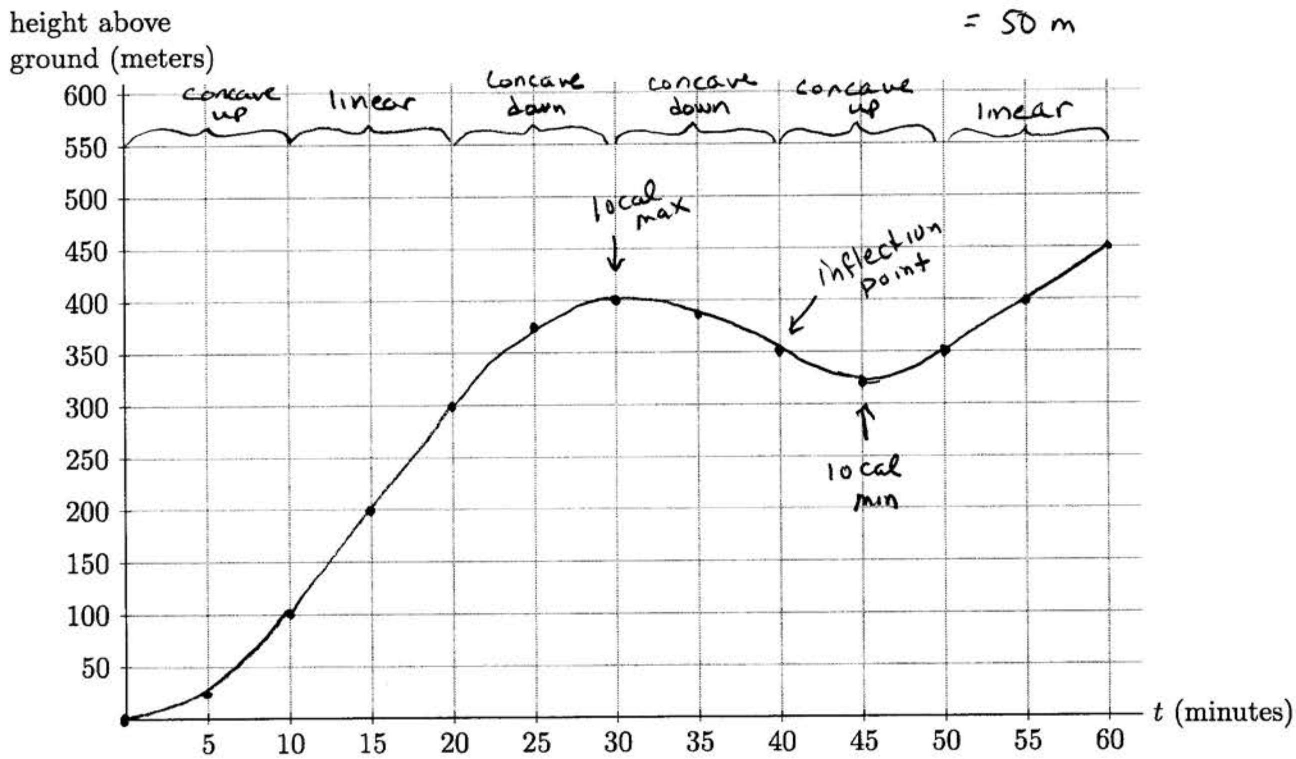
2. [12 points] Erna is a rock climber who is climbing a rock formation. Due to the shape of the formation, the rate at which Erna ascends varies and is given by the graph below.



a. [10 points] Erna starts climbing from the ground at $t = 0$, and she reaches the top at $t = 60$, where t is measured in minutes. Use the axes below to carefully sketch a graph of her height above the ground for $0 \leq t \leq 60$.

- Clearly indicate the coordinates of the points on your graph at $t = 0$, 10, 20, 30, 40, 50, and 60.
- Be sure that local extrema and concavity are clear.

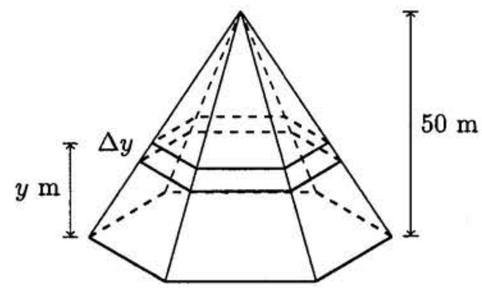
1 Box = (5 min)(10 m/min)
= 50 m



b. [2 points] How tall was the rock formation that Erna climbed?

Answer: Height of Rock Formation = 450 meters

3. [8 points]
Consider a tent that is 50 meters tall whose base is a regular hexagon (i.e. a 6-sided polygon with equal length sides and equal angles) and whose horizontal cross-sections are also regular hexagons.
(See figure on the right.)
Suppose the perimeter of the base is 72 meters.
Let $P(y)$ be the perimeter, in meters, of a horizontal cross section y meters above the ground.



a. [2 points] It turns out that $P(y)$ is a linear function of the variable y. (You do not need to verify this.) Find a formula for $P(y)$.

y	$P(y)$
0	72
50	0

Slope $= \frac{0-72}{50-0} = -1.44$

Answer: $P(y) =$ $72 - 1.44y$

b. [3 points] The area of a regular hexagon with perimeter p is equal to $\frac{\sqrt{3}}{24}p^2$.

Write an expression that gives the approximate volume, in cubic meters, of a horizontal slice of the region inside the tent that is Δy meters thick and y meters above the ground. (Assume here that Δy is small but positive.) Your expression should <u>not</u> involve any integrals.

area of slice $= \frac{\sqrt{3}}{24} P(y)^2 = \frac{\sqrt{3}}{24}(72-1.44y)^2$

Volume of slice $= \frac{\sqrt{3}}{24}(72-1.44y)^2 \Delta y$

Answer: Volume of slice $\approx$ $\frac{\sqrt{3}}{24}(72-1.44y)^2 \Delta y$

c. [3 points] Write, but do <u>not</u> evaluate, an expression involving one or more integrals that gives the total volume, in cubic meters, inside the tent.

Answer: Volume $=$ $\boxed{\int_0^{50} \frac{\sqrt{3}}{24}(72-1.44y)^2\, dy} = 3600\sqrt{3}\ m^3$

4. [12 points] For each of the questions below, circle <u>all</u> of the available correct answers.
Circle "NONE OF THESE" if none of the available choices are correct.
You must circle at least one choice to receive any credit.
No credit will be awarded for unclear markings. No justification is necessary.

a. [3 points] Suppose f and g are continuous functions defined for all real numbers. Which of the following <u>must</u> be true?

i. The average value of the sum of f and g over the interval $[-5,5]$ is equal to the sum of the average value of f over $[-5,5]$ and the average value of g over $[-5,5]$.

ii. $\int_{-5}^{5}(3+f(x))\,dx = 3+\int_{-5}^{5} f(x)\,dx$

iii. $\int f'(x^2)\,dx = \dfrac{f(x^2)}{2x}+C$

iv. $\int_{-5}^{5} f(x)\cdot g(x)\,dx = \int_{-5}^{5} f(x)\,dx \cdot \int_{-5}^{5} g(x)\,dx$

v. NONE OF THESE

Unless you know something about W (e.g. increasing, decreasing, concave up or concave down), all Riemann sum estimates could be way off, in either direction.

b. [3 points] Suppose that W is a function that is continuous and positive on the interval $[0,1]$. Consider the the approximations RIGHT(100), LEFT(100), and MID(100) of the definite integral $\int_0^1 W(x)\,dx$. Which of the following <u>must</u> be true?

i. $\text{LEFT}(100) \le \int_0^1 W(x)\,dx$

ii. $\text{RIGHT}(100) \le \int_0^1 W(x)\,dx$

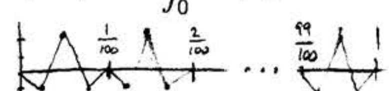
iii. $\text{LEFT}(100) \ge \int_0^1 W(x)\,dx$

iv. $\text{RIGHT}(100) \ge \int_0^1 W(x)\,dx$

v. MID(100) is at least as close to $\int_0^1 W(x)\,dx$ as RIGHT(100) is to $\int_0^1 W(x)\,dx$.

vi. NONE OF THESE

Counterexample :



$\text{RIGHT}(100) = \int_0^1 W(x)\,dx = 0$, but $\text{MID}(100) > 0$.

c. [3 points] Let $Q(x) = \int_1^x \ln(t)\,dt$. Then which of the following <u>must</u> be true?

i. $x = \dfrac{d}{dx}\left[\int_1^{e^x} \ln(t)\,dt\right]$

ii. $\dfrac{d}{dp}\left[Q(4+\sin(p))\right] = \cos(p)\left[\ln(4+\sin(p))\right]$

iii. $Q'(x) = \ln(x)$

iv. $\dfrac{d}{dr}\left[\int_1^{1+r^2} \ln(t)\,dt\right] = \dfrac{d}{dr}\left[\int_1^{r} \ln(t^2+1)\,dt\right]$

v. NONE OF THESE

d. [3 points] Let $g(x)$ be a differentiable function that is decreasing and concave up on the interval $[0,1]$. Suppose $G(x)$ is an antiderivative of $g(x)$. Which of the following <u>must</u> be true?

i. $G(x) \le 0$ on $[0,1]$.

ii. $G(x)$ is increasing on $[0,1]$.

iii. $G(x)$ is concave down on $(0,1)$.

iv. $G(x)$ has no inflection points in $(0,1)$.

v. NONE OF THESE

5. [7 points] Lovise is a character in a video game. As part of the game she lifts a crate straight up from the ground to a height of 60 m. Gravity in this video game world is not constant! At a height of h meters above the ground, the acceleration due to gravity is $7e^{-h}$ meters per second per second. The crate has a mass of 5 kg.

Recall: Weight is the force exerted by gravity and is equal to mass times acceleration due to gravity.

a. [3 points] Write an expression that approximates the amount of work done by Lovise in the video game to lift the crate from a height of h meters above the ground to $h + \Delta h$ meters above the ground. (Assume here that Δh is positive but very small.)
Your expression should <u>not</u> involve any integrals.

Mass = 5kg
acceleration = $7e^{-h}$ m/s²
weight = $(5\text{kg})(7e^{-h}\ \text{m/s}^2) = 35e^{-h}$ N
distance to lift = Δh m
Work = (weight)(distance) = $35e^{-h}\Delta h$ Joules

Answer: Work ≈ $35e^{-h}\Delta h$ joules

b. [4 points] Write and evaluate an integral that gives the total work done by the character Lovise in lifting the crate to a height of 60m above the ground.
(You may do this by hand or by using your calculator. Give an exact answer or round your answer to two decimal places.)

$$\int_0^{60} 35e^{-h}\,dh = -35e^{-h}\Big|_0^{60} = -35e^{-60} + 35e^{-0}$$

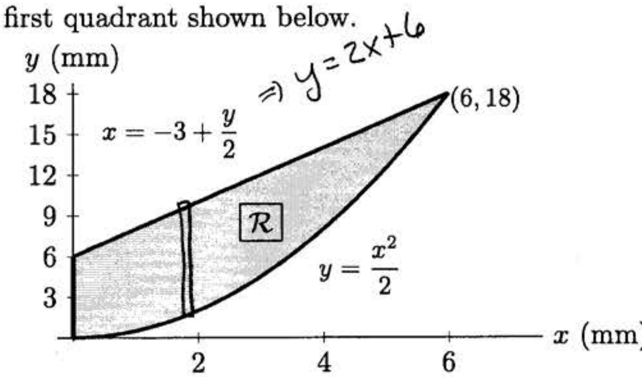
Answer: Integral Expression: $\int_0^{60} 35e^{-h}\,dh$

Numerical Final Answer (with units): $35(1-e^{-60})$ joules

6. [12 points] Let $\mathcal{R}$ be the shaded region in the first quadrant shown below.
The region $\mathcal{R}$ is bounded by:

- the y-axis,
- the graph of $y = \frac{x^2}{2}$, and
- the graph of $x = -3 + \frac{y}{2}$. $\boxed{y = 2x+6}$

The units on both axes are millimeters (mm).



a. [4 points] Write, but do NOT evaluate, an expression involving one or more integrals that gives the volume, in mm^3, of the solid whose base is the region $\mathcal{R}$ and whose cross-sections perpendicular to the x-axis are squares.

Side of slice $= (2x+6) - \frac{x^2}{2} = -\frac{1}{2}x^2 + 2x + 6$

area of slice $= \left(-\frac{1}{2}x^2 + 2x + 6\right)^2$

Volume of slice $= \left(-\frac{1}{2}x^2 + 2x + 6\right)^2 \Delta x$

Answer: Volume = $\int_0^6 \left(-\frac{1}{2}x^2 + 2x + 6\right)^2 dx$

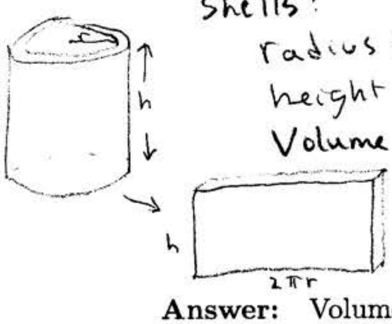
b. [4 points] Write, but do NOT evaluate, an expression involving one or more integrals that gives the volume, in mm^3, of the solid formed by rotating the region $\mathcal{R}$ around the y-axis.

Shells:

radius of shell $= x$

height of shell $= -\frac{1}{2}x^2 + 2x + 6$

Volume of shell $= 2\pi r h \Delta x = 2\pi x\left(-\frac{1}{2}x^2 + 2x + 6\right)\Delta x$



Answer: Volume = $\int_0^6 2\pi x \left(-\frac{1}{2}x^2 + 2x + 6\right) dx$

c. [4 points] Write, but do NOT evaluate, an expression involving one or more integrals that gives the mass, in grams, of a thin plate in the shape of the region $\mathcal{R}$ that has mass density given by $\delta(x) = (1 + x^2)\ \text{g/mm}^2$.

Area of slice $= \left(-\frac{1}{2}x^2 + 2x + 6\right)\Delta x$

mass of slice $=$ (Area)(density) $= \left(-\frac{1}{2}x^2 + 2x + 6\right)(1 + x^2)\Delta x$

Answer: Mass = $\int_0^6 \left(-\frac{1}{2}x^2 + 2x + 6\right)(1 + x^2)\, dx$

7. [12 points] Note that the problems on this page do not depend on each other.

a. [4 points] Suppose $F(x)$ is an antiderivative of $f(x) = e^{-x^2}$ such that $F(2) = 10$. Write an integral expression for the function $F(x)$. (Your expression should not involve the letters f or F.) Remember to be careful with notation.

Answer: $F(x) =$ $10 + \int_2^x e^{-t^2}\,dt$

b. [4 points] Suppose $H(x)$ is an antiderivative of $h(x) = \sin(x^2)$. Write an expression for the average value of $h(x)$ on the interval $[-1, 1]$. Your expression should not involve any integrals but may involve function names.

avg Value $= \frac{1}{1-(-1)} \int_{-1}^{1} h(x)\,dx$

Answer: Average Value $=$ $\frac{1}{2}\left[H(1) - H(-1)\right]$

c. [4 points] Suppose $G(x)$ is an antiderivative of $g(x) = \sqrt{x^4 - 1}$ for $x > 1$. Find the arc length of the graph of $G(x)$ from $x = 2$ to $x = 3$. Show your work. You may use your calculator to evaluate any integrals. Give the exact answer or round to two decimal places.

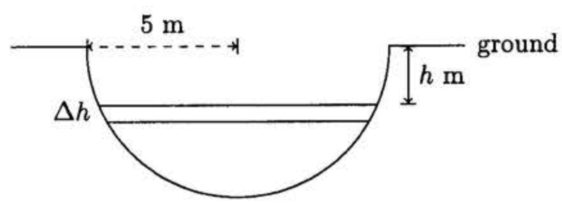
$$\text{arclen} = \int_a^b \sqrt{1 + G'(x)^2}\,dx = \int_2^3 \sqrt{1 + (x^4 - 1)}\,dx$$

$$= \int_2^3 x^2\,dx = \frac{1}{3}x^3\Big|_2^3 = \frac{1}{3}\left[3^3 - 2^3\right]$$

Answer: Arc Length $=$ $19/3$

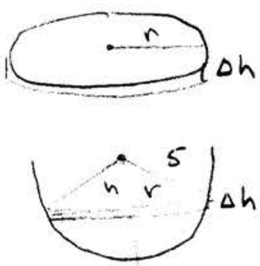
8. [12 points]

Alicia is building a pond in her backyard. The pond will be in the shape of hemisphere with radius 5 meters. A side view of the hole for the pond is shown in the figure on the right.
Note: The curved region shown is a semicircle of radius 5 meters, and cross-sections of the hole parallel to the ground are circles.



Alicia discovers that the density (in kg/m^3) of the dirt in her yard is given by the function $\rho(h) = 1.5 + (h-1)^3$ where h is distance (in meters) below ground. In this problem, you may assume the acceleration due to gravity is $g = 9.8$ m/s^2.

a. [4 points] Write an expression that gives the approximate mass of a horizontal slice of dirt with thickness Δh meters that is h meters below the ground. See diagram. (Assume that Δh is small but positive.) Your expression should not involve any integrals.



radius of slice $= \sqrt{25-h^2}$ m

volume of slice $= \pi r^2 \Delta h = \pi(25-h^2)\Delta h$ m^3

density of slice $= \rho(h) = 1.5 + (h-1)^3$ kg/m^3

mass of slice $=$ (volume)(density) $=$

$r^2 + h^2 = 5^2$

Answer: Mass of slice $\approx$ $\pi(25-h^2)(1.5+(h-1)^3)\Delta h$ kg

b. [3 points] Write, but do not evaluate, an expression involving one or more integrals that gives the mass (in kg) of the dirt Alicia removes in order to create the hole for her pond.

Answer: Mass $=$ $\int_{h=0}^{5} \pi(25-h^2)(1.5+(h-1)^3)\,dh$ kg

c. [5 points] As Alicia digs, she lifts the dirt 1 meter above the ground to put it into the back of a truck. Write, but do not evaluate, an expression involving one or more integrals that gives the work Alicia does to remove all the dirt from the hole for her pond.

Weight of slice $=$ (mass)(acceleration due to gravity)

$= 9.8\pi(25-h^2)(1.5+(h-1)^3)\Delta h$ N

dist to lift slice $= h+1$ m

work to lift slice $=$ (weight)(distance)

Answer: Work $=$ $\int_0^5 9.8\pi(h+1)(25-h^2)(1.5+(h-1)^3)\,dh$ joules (include units)

9. [13 points] Simba is currently practicing his tennis swing by hitting a ball against a wall. During his first attempt he only managed to hit the ball 5 times. Each time he practices, he manages to hit the ball more so that in his nth attempt he hits the ball $2n^2 + 2n + 1$ times. After 10 attempts he stops and makes the following claim:

Simba's Claim: I must have the hit ball at least 800 times!

The total number of hits in his first 10 attempts is given by the following sum:

$$H = \sum_{n=1}^{10} (2n^2 + 2n + 1)$$

a. [3 points] Find a definite integral for which RIGHT(10) $= H$.

if $\Delta x = 1$,

$$H = \sum_{n=1}^{10} (2x^2 + 2x + 1)\Delta x \text{ , so}$$

Answer: $\int_0^{10} (2x^2 + 2x + 1)\, dx$

b. [3 points] Using only your integral from part (a), is it possible to evaluate Simba's claim? If so, evaluate his claim using this integral.
Note: To earn credit, clear justification based on your integral must be provided for the answer you circle.

Answer: Based only on your integral from part (a), we can conclude that Simba

Circle your answer.

Hit the ball at least 800 times   Did not hit the ball 800 times   Not possible to tell

Justification:

Since $2x^2 + 2x + 1$ is increasing on $[0,10]$ (because its derivative, $4x+2$, is positive),

$$\text{RIGHT}(10) \geq \int_0^{10} (2x^2 + 2x + 1)\,dx$$

$$= \frac{2}{3}x^3 + x^2 + x \Big|_0^{10} = \left[\left(\frac{2}{3}(1000) + 100 + 10\right) - 0\right] = 776\tfrac{2}{3}$$

We know he hit it at least 777 times, but he might not have made it to 800.

This problem continues on the next page.

This is a continuation of the problem from the previous page.
For your convenience, a reminder of the setup is included here.

Simba is currently practicing his tennis swing by hitting a ball against a wall. After 10 attempts he stops and makes the following claim:

Simba's Claim: I must have the hit ball at least 800 times!

The total number of hits in his first 10 attempts is given by: $H = \sum_{n=1}^{10} (2n^2 + 2n + 1)$.

c. [4 points]Note that $\sum_{n=1}^{10} (2n^2 + 2n + 1) = 2 \cdot \sum_{n=1}^{10} \left[\frac{1}{2}\left(n^2 + (n+1)^2\right)\right]$.

(You do not need to verify this.)

Use the resulting formula $H = 2 \cdot \sum_{n=1}^{10} \left[\frac{1}{2}\left(n^2 + (n+1)^2\right)\right]$

to find a definite integral for which TRAP(10) $= H$.

Answer: $\int_1^{11} 2x^2\,dx$

d. [3 points] Using only your integral from part (c) is it possible to evaluate Simba's claim? If so, evaluate his claim using this integral.
Note: To earn credit, clear justification based on your integral must be provided for the answer you circle.

Answer: Based only on your integral from part (c), we can conclude that Simba

Circle your answer.

$\boxed{\text{Hit the ball at least 800 times}}$ Did not hit the ball 800 times Not possible to tell

Justification:

Because $2x^2$ is concave up, TRAP(10) is an over estimate for the integral, which means the integral is an underestimate of the sum. The integral is

$$\int_1^{11} 2x^2\,dx = \frac{2}{3}x^3\Big|_1^{11} = \frac{2}{3}(11)^3 - \frac{2}{3}(1)^3 = 886\tfrac{2}{3}.$$

So the sum is at least that much.

(Actual value of sum is 890. TRAP is much closer than RIGHT!)