On my honor, as a student,
I have neither given nor received
unauthorized aid on this academic work. **Initials:** ______

Do not write in this area

Math 116 — Second Midterm — November 12, 2018

Your Initials Only: ______ Your U-M ID # (not uniqname): ______________

Instructor Name: ______________________ Section #: ________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. Do not separate the pages.
   If pages do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
4. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is <u>not</u> permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. The density function of a normal distribution with mean μ and standard deviation $\sigma > 0$ is
    $$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}.$$
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	9	
3	9	
4	8	
5	10	

Problem	Points	Score
6	10	
7	12	
8	10	
9	9	
10	12	
Total	100	

1. [11 points] For each of parts **a - d** below:
   - Find the *exact* value, if possible. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
   - If the given limit or integral either does not exist or diverges, write "DOES NOT EXIST".
   - If there is not enough information, write "NOT ENOUGH INFO".
   - You do not have to show work, but work shown might be considered for partial credit.

a. [2 points] Suppose $f(x)$ is a continuous, positive, and decreasing function such that $\int_2^{\infty} f(x)\,dx$ converges. Find $\lim_{x\to\infty} f(x)$.

Answer: $\lim_{x\to\infty} f(x) =$ 0

b. [3 points] Find $\int_0^{\infty} \frac{1}{x^{0.7}}\,dx$. $= \lim_{b\to\infty} \int_0^b x^{-.7}\,dx = \lim_{b\to\infty} \frac{1}{.3} x^{.3} \Big|_0^b = \lim_{b\to\infty} \frac{b^{.3} - 0^{.3}}{.3}$

$= \infty$

Answer: $\int_0^{\infty} \frac{1}{x^{0.7}}\,dx =$ DOES NOT EXIST

c. [3 points] Suppose μ is a real number. Find $\int_{-\infty}^{\infty} e^{-(x-\mu)^2/0.0002}\,dx$.

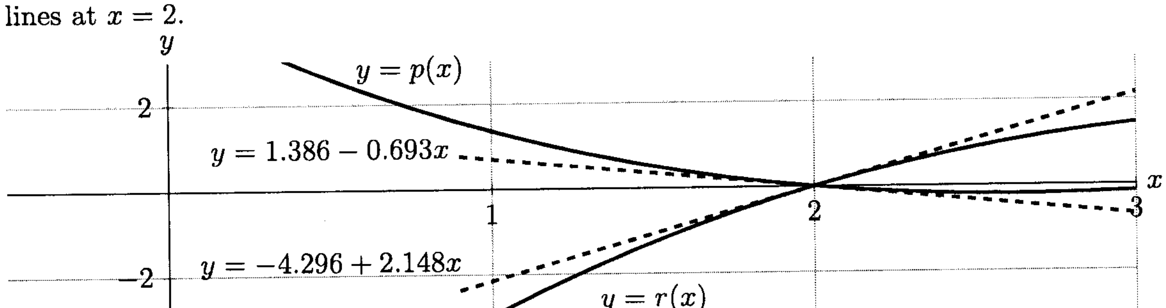
Your answer may involve μ.

$-\frac{(x-\mu)^2}{.0002} = -\frac{(x-\mu)^2}{\frac{2}{10000}} = -\frac{1}{2}(100)^2(x-\mu)^2 = -\frac{1}{2}(100x - 100\mu)$.

So let $w = 100x - 100\mu \Rightarrow dw = 100\,dx$

Answer: $\int_{-\infty}^{\infty} e^{-(x-\mu)^2/0.0002}\,dx =$ $\int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2} \cdot \frac{dw}{100} = \boxed{\frac{\sqrt{2\pi}}{100}}$

d. [3 points] The graph below shows two functions $p(x)$ and $r(x)$, as well as their tangent lines at $x = 2$.



Find the value of $\lim_{x\to 2} \frac{p(x)}{r(x)}$. $= \lim_{x\to 2} \frac{p'(x)}{g'(x)} = \frac{-.693}{2.148}$

by L'Hopital's Rule, since $\frac{0}{0}$

Answer: $\lim_{x\to 2} \frac{p(x)}{r(x)} =$ $\boxed{-\frac{.693}{2.148}}$

2. [9 points] Note: *"Closed form" here means that the expression should* NOT *include sigma notation or ellipses (...) and should* NOT *be recursive.*

Michel is studying how the mass of a certain collection of bacterial cells behaves in the presence of a parasite. He notices that from noon to midnight of each day, the parasite eats 60% of the mass of the bacterial cells. Then the parasite sleeps until noon the next day. While the parasite sleeps, the remaining 40% of the collection of bacterial cells doubles in mass.

At noon on the first day, the mass of the collection of bacterial cells is 100 grams.

a. [3 points] Let X_n be the mass, in grams, of bacterial cells present at noon on day n. Note that $X_1 = 100$. Calculate X_2 and X_3, and find a closed form expression for X_n.

n	X_n	Uneaten	After regrowth
1	100	$(.4)(100)$	$2(.4)(100) = 80 = (.8)(100)$
2	80	$(.4)(80)$	$2(.4)(80) = 64 = (.8)^2(100)$
3	64		

Answer: $X_2 =$ 80 g and $X_3 =$ 64 g

Answer: $X_n = 100(.8)^{n-1}$ g

b. [4 points] Let K_n be the total mass, in grams, of bacterial cells that the parasite has consumed in the first n days. For example, on day 1 the parasite consumes 60% of 100 grams, which is 60 grams, so $K_1 = 60$. Calculate K_2 and K_3, and find a closed form expression for K_n.

n	K_n
1	60
2	$60 + (.6)(80) = 108$
3	$108 + (.6)(64) = 146.4$

Amount eaten on day i is $(.6)X_i = (.6)(100)(.8)^{i-1} = 60(.8)^{i-1}$

so $K_n = \sum_{i=1}^{n} 60(.8)^{i-1} = 60\left(1 + (.8) + (.8)^2 + \cdots + (.8)^{n-1}\right) = 60\frac{1-(.8)^n}{1-.8}$

$= \frac{60}{.2}\left(1-(.8)^n\right) = 300\left(1-(.8)^n\right)$

Answer: $K_2 =$ 108 g and $K_3 =$ 146.4 g

Answer: $K_n = 300\left(1-(.8)^n\right)$ g

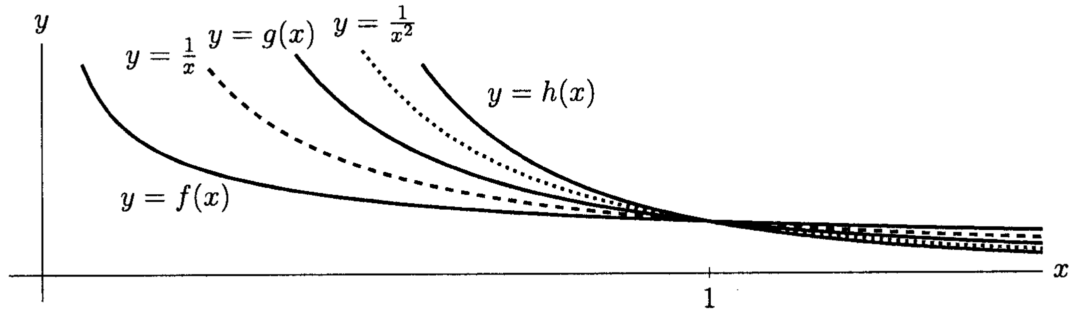
c. [2 points] If this continued forever, how many grams of bacterial cells would the parasite eventually eat?

$$\lim_{n\to\infty} 300\left(1-(.8)^n\right) = 300$$

Answer: Mass = 300 g

3. [9 points] For each of the questions on this page you must circle exactly *one choice* to receive any credit. No credit will be awarded for unclear markings and no justification is necessary.

a. [3 points] The functions shown below are continuous on the interval $(0, 2)$ and have a vertical asymptote at $x = 0$. The only point of intersection between the graphs of any of the functions is at $x = 1$, where all 5 graphs intersect.



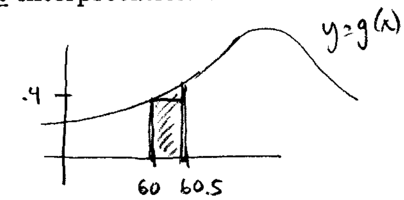
Determine whether the following integrals converge or diverge. If there is not enough information to determine convergence or divergence, circle "NOT ENOUGH INFO".

i. $\int_0^1 f(x)\,dx$ A. CONVERGES B. DIVERGES C. NOT ENOUGH INFO

ii. $\int_0^1 g(x)\,dx$ A. CONVERGES B. DIVERGES C. NOT ENOUGH INFO

iii. $\int_0^1 h(x)\,dx$ A. CONVERGES B. DIVERGES C. NOT ENOUGH INFO

b. [3 points] Suppose $g(x)$ is the probability density function for the height, in inches, of a certain population of people. Which of the following is the <u>one best</u> interpretation of the equation $g(60) = 0.4$?



A. About 40% of this population is exactly 60 inches tall.

B. About 40% of this population is at most 60 inches tall.

C. About 60% of this population is at most 60 inches tall.

D. About 40% of this population is between 60 and 60.4 inches tall.

E. About 20% of this population is between 60 and 60.5 inches tall.

F. About 60% of this population is between 60 and 61 inches tall.

$P(60 \le \text{height} \le 60.5)$
$= \int_{60}^{60.5} g(x)\,dx$
$\approx$ area of left rectangle
$= (.4)(.5) = .2$

c. [3 points] Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{-1}{1+2n}\, x^n$.

For each of the values of x below, does the power series converge or diverge?

i. At $x = 1$, the power series A. CONVERGES B. DIVERGES

ii. At $x = 0$, the power series A. CONVERGES B. DIVERGES

iii. At $x = -2$, the power series A. CONVERGES B. DIVERGES

4. [8 points] Determine whether the following series converge or diverge.
Fully justify your answer. Show all work and include any convergence tests used.

a. [4 points] $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sin\left(\frac{1}{n}\right)}$

Circle one: **Converges** $\boxed{\textbf{Diverges}}$

Justification:

As $n \to \infty$, $\frac{1}{n} \to 0$, so $\sin\left(\frac{1}{n}\right) \to \sin(0) = 0$

which means $\frac{1}{\sin\left(\frac{1}{n}\right)}$ does not approach 0.

Since the terms of the series do not approach 0,

the series diverges

b. [4 points] $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n^2+3^n}$

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

Ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)^2+3^{n+1}} \cdot \frac{n^2+3^n}{2^n} \right|$$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n^2+3^n}{(n+1)^2+3^{n+1}} = 2 \lim_{n\to\infty} \frac{n^2+3^n}{(n+1)^2+3^{n+1}}$$

The exponentials dominate the polynomials, so

that's $2 \lim_{n\to\infty} \frac{3^n}{3^{n+1}} = \frac{2}{3}$, which is less than 1.

So converges by the ratio test.

5. [10 points] You are at a bus stop waiting for a bus to arrive. The cumulative distribution function for the time, in minutes, a passenger will wait for the next bus to arrive is given by

$$P(t) = \begin{cases} 0 & t \le 0 \\ 1 - e^{-0.05t} & t > 0. \end{cases}$$

a. [3 points] What is the median amount of time that a passenger has to wait for a bus to arrive? Provide an **exact** answer. Remember to show all your work.

If T is the median time, then

$\frac{1}{2} = \text{Prob}(\text{wait} < T) = P(T) = 1 - e^{-.05T} \Rightarrow e^{-.05T} = \frac{1}{2}$

Answer: Median = $\boxed{\frac{\ln(.5)}{-.05}} = 20 \ln 2$

You decide that you are going to take the 2nd bus that arrives. It can be shown that the number of minutes a passenger has to wait for 2 buses to arrive has probability density function

$$q(t) = \begin{cases} 0 & t \le 0 \\ Cte^{-0.05t} & t > 0 \end{cases}$$

for some constant C.

b. [5 points] Find the value of C. **Show all your work using correct notation.** Any evaluation of integrals must be done **without a calculator.**

Let $w = .05t \Rightarrow t = 20w$
$dt = 20dw$
$t = 0 \Rightarrow w = 0$
$t \to \infty \Rightarrow w \to \infty$

$1 = \int_{-\infty}^{\infty} q(t)\,dt = \int_0^{\infty} Cte^{-.05t}\,dt$

$= C\int_0^{\infty} (20w)e^{-w}(20dw) = 400C\int_0^{\infty} we^{-w}\,dw$

Let $u = w$ $v' = e^{-w}$
$u' = 1$ $v = -e^{-w}$

$= 400C \lim_{b\to\infty} \left[-we^{-w}\Big|_0^b - \int_0^b -e^{-w}\,dw \right]$

$= 400C \lim_{b\to\infty} \left[\frac{-b}{e^b} - e^{-w}\Big|_0^b \right] = 400C \lim_{b\to\infty} \left[\cancel{\frac{-b}{e^b}}^{\,0} + e^{-0} \right]$

$= 400C.$

Answer: $C = 1/400 = .0025$

c. [2 points] Write an expression (possibly involving one or more integrals) for the mean number of minutes it takes for 2 buses to arrive. (You do not need to evaluate your expression.)

$\int_{-\infty}^{\infty} t\,q(t)\,dt$

Answer: Mean = $\frac{1}{400}\int_0^{\infty} t^2 e^{-.05t}\,dt$

6. [10 points] Consider the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\,5^n}\,(x+3)^n.$$

a. [2 points] What is the center of the interval of convergence of this power series?

Answer: The center is at $x =$ -3

For parts b and c below, show every step of any calculations and fully justify your answer with careful reasoning.

b. [3 points] Find the radius of convergence of this power series.

Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x+3)^{n+1}}{(n+1)\,5^{n+1}} \cdot \frac{n\,5^n}{(-1)^n(x+3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{(x+3)^{n+1}}{(x+3)^n}\cdot\frac{\cancel{n}^{\,1}}{\cancel{n+1}}\cdot\frac{5^n}{5^{n+1}}\right| = \frac{|x+3|}{5}$$

Converges when ratio < 1, i.e.

$$\frac{|x+3|}{5} < 1 \Rightarrow |x+3| < 5$$

Answer: Radius of Convergence: 5

c. [5 points] Find the interval of convergence for this power series.

Converges if $|x+3| < 5$ and diverges if $|x+3| > 5$.

On the boundary:

$\underline{x+3 = 5} \Rightarrow x = 2 \Rightarrow$ Series $= \sum \frac{(-1)^n}{n5^n}\cdot 5^n = \sum \frac{(-1)^n}{n}$

which converges by Alternating Series Test, since signs alternate, |terms| decreases, and terms $\to 0$.

$\underline{x+3 = -5} \Rightarrow x = -8 \Rightarrow$ series $= \sum \frac{(-1)^n}{n5^n}(-5)^n = \sum \frac{1}{n}$

which diverges by the p-test ($p=1$).

Answer: Interval of Convergence: $(-8, 2]$

7. [12 points] For each of the questions on this page:
You must circle at least one choice to receive any credit.
No credit will be awarded for unclear markings. No justification is necessary.

For parts **a - c** below, circle all of the available correct answers, and circle "NONE OF THESE" if none of the available options are correct.

a. [3 points] Let a_n be a sequence of positive numbers, and let $S_n = a_1 + a_2 + \cdots + a_n$. Suppose $\lim_{n\to\infty} \frac{S_n}{n^2} = 2$. Which of the following must be true?

Suppose $a_n = 4n-2$. Then the sequence a_n and the series $\sum a_n$ both diverge, even though $S_n = 2+6+10+\cdots+(4n-2) = 2n^2$. So that's a counterexample for (i) and (ii)

i. The sequence a_n converges.

ii. The sequence S_n diverges.

$S_n \approx 2n^2$ for large n

iii. The series $\sum_{n=1}^{\infty} a_n$ converges.

iv. The series $\sum_{n=1}^{\infty} S_n$ diverges.

$S_n \geq a_1 > 0$ so terms don't go to 0

v. The series $\sum_{n=1}^{\infty} \frac{1}{S_n}$ converges.

vi. NONE OF THESE

Limit compare with $\sum \frac{1}{n^2}$

b. [3 points] Which of the following series are conditionally convergent?

i. $\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n$

ii. $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

iii. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$

iv. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

v. $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{4n^3+5}$

vi. NONE OF THESE

(i), (ii), (iii) are absolutely convergent
(v) is divergent

c. [3 points] Suppose $f(x)$ is a positive, decreasing function on $[0, \infty)$ and suppose $\sum_{n=0}^{\infty} f(n) = 3$. Let $B_n = \int_0^n f(x)\,dx$ for $n \geq 0$, . Which of the following must be true?

i. $\lim_{n\to\infty} f(n) = 0$

ii. $\lim_{n\to\infty} f(n) = 3$

iii. $\int_0^{\infty} f(x)\,dx = 3$

iv. $\sum_{n=0}^{\infty} (-1)^n f(n)$ converges

by AST

v. The sequence B_n is bounded and increasing.

vi. NONE OF THESE

increasing because $f(x) > 0$, bounded because $\int_0^{\infty} f(x)dx$ converges by integral test

8. [10 points] Determine whether the following improper integrals converge or diverge. Show all of your work and indicate any theorems you used to conclude convergence or divergence of the integrals. Any direct evaluation of integrals must be done without using a calculator.

a. [5 points] $\displaystyle\int_3^\infty \frac{\ln(x)}{x^2}\,dx$

Circle one: **Converges** **Diverges**

Justification:

Let $u = \ln x$, $v' = x^{-2}$

$u' = x^{-1}$ $v = -x^{-1}$

$$\int_3^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_3^b uv'\,dx = \lim_{b\to\infty}\left[uv\Big|_3^b - \int_3^b u'v\,dx\right]$$

$$= \lim_{b\to\infty}\left[-\frac{\ln x}{x}\Big|_3^b - \int_3^b -x^{-2}\,dx\right] = \lim_{b\to\infty}\left[-\frac{\ln b}{b} + \frac{\ln 3}{3} - x^{-1}\Big|_3^b\right]$$

$$= \lim_{b\to\infty} \frac{-\ln b}{b} + \frac{\ln 3}{3} - \frac{1}{b} + \frac{1}{3} = \frac{1+\ln 3}{3} - \lim_{b\to\infty}\frac{1+\ln b}{b}$$

By L'Hôpital's Rule, that final limit is $\lim_{b\to\infty}\frac{1/b}{1} = 0$.

So converges to $\frac{1+\ln 3}{3}$

b. [5 points] $\displaystyle\int_0^\infty \frac{3}{4x^2+5\sqrt{x}}\,dx$

Circle one: **Converges** **Diverges**

Justification:

on $(0,1)$, $\frac{3}{4x^2+5\sqrt{x}} < \frac{3}{5\sqrt{x}} = \frac{3}{5}\cdot\frac{1}{x^{1/2}}$. $\int_0^1 \frac{3}{5}\cdot\frac{1}{x^{1/2}}\,dx$ converges by the p-test $(p=\frac{1}{2})$, so $\int_0^1 \frac{3}{4x^2+5\sqrt{x}}\,dx$ converges by comparison

on $(1,\infty)$, $\frac{3}{4x^2+5\sqrt{x}} < \frac{3}{4x^2} = \frac{3}{4}\cdot\frac{1}{x^2}$. $\int_1^\infty \frac{3}{4}\frac{1}{x^2}$ converges by the p-test $(p=2)$, so $\int_1^\infty \frac{3}{4x^2+5\sqrt{x}}$ converges by comparison.

$$\therefore \int_0^\infty \frac{3}{4x^2+5\sqrt{x}}\,dx = \int_0^1 + \int_1^\infty \text{ converges.}$$

9. [9 points] Consider the function $f(x) = \begin{cases} \dfrac{\sin(x^2)}{x} & x \neq 0 \\ c & x = 0. \end{cases}$

a. [2 points] Find the value of c that makes the function $f(x)$ continuous at $x = 0$. Show your work carefully.

$$\lim_{x\to 0} \frac{\sin(x^2)}{x} \underset{\text{by L'Hôpital}}{=} \lim_{x\to 0} \frac{2x\cos(x^2)}{1} = 2\cdot 0\cdot 1 = 0$$

Answer: $c =$ 0

b. [2 points] Let b_n be the nth positive value of x such that $f(x) = 0$. Write a formula for b_n.

$f(x) = 0 \Rightarrow \dfrac{\sin(x^2)}{x} = 0 \Rightarrow \sin(x^2) = 0 \Rightarrow x^2 = n\pi$

for some integer n.

Answer: $b_n =$ $\sqrt{n\pi}$

For parts **c** and **d** below, let $a_n = \displaystyle\int_{b_{n-1}}^{b_n} f(x)\,dx$ for $n \geq 1$.

c. [2 points] Explain why a_n is an alternating sequence.

Because sin alternates from + to − between its zeroes:

$$a_n = \int_{\sqrt{(n-1)\pi}}^{\sqrt{n\pi}} \frac{\sin(x^2)}{x}\,dx \quad \text{Let } w = x^2, \Rightarrow dw = 2x\,dx$$

$$= \int_{(n-1)\pi}^{n\pi} \frac{\sin(w)}{2w}\,dw$$

d. [3 points] Compute $\lim_{n\to\infty} a_n$. Provide clear justification and show your work.

$$|a_n| = \left| \int_{(n-1)\pi}^{n\pi} \frac{\sin(w)}{2w}\,dw \right| \leq \int_{(n-1)\pi}^{n\pi} \left|\frac{\sin(w)}{2w}\right| dw \leq \int_{(n-1)\pi}^{n\pi} \frac{dw}{2w}$$

$$= \frac{1}{2}\ln w \Big|_{(n-1)\pi}^{n\pi} = \frac{1}{2}\left[\ln(n\pi) - \ln((n-1)\pi)\right] = \frac{1}{2}\ln\left(\frac{n}{n-1}\right).$$

as $n\to\infty$, $\dfrac{n}{n-1} \to 1$, so $|a_n| \to 0$, so $a_n \to 0$

Answer: $\lim_{n\to\infty} a_n =$ 0

10. [12 points] Provide an example for each of the following. Note that there are examples in each case.

a. [3 points] A sequence a_n that is bounded but not monotonic.

Answer: $a_n =$ $(-1)^n$

b. [3 points] A sequence b_n such that $\sum_{n=1}^{\infty} b_n$ converges but $\sum_{n=1}^{\infty} b_n^2$ diverges.

Answer: $b_n =$ $\dfrac{(-1)^n}{\sqrt{n}}$

c. [3 points] A sequence c_n and a function $g(x)$ such that $g(n) = c_n$ for all $n \geq 1$, the improper integral $\int_1^{\infty} g(x)\,dx$ diverges, and the series $\sum_{n=1}^{\infty} c_n$ converges.

Note: You may describe your function $g(x)$ by giving either a formula or a well-drawn and clearly labeled graph.

Must violate one of the conditions of the integral test, so either not positive or not decreasing.

Answer: $c_n =$ 0 and $g(x) =$ $\sin(\pi x)$

d. [3 points] A sequence d_n with $d_n \geq 0$ for $n \geq 1$ such that

$$\lim_{n\to\infty} d_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^n d_n \text{ diverges.}$$

Hint: Consider defining d_n piecewise, with one formula for when n is odd and one for when n is even.

Since AST doesn't apply even though terms alternate and $\to 0$, must be the case that terms don't decrease in magnitude.

Answer: $d_n =$ $\begin{cases} 2/n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

So $\sum_{1}^{\infty}(-1)^n d_n = -0 + \frac{2}{2} - 0 + \frac{2}{4} - 0 + \frac{2}{6} - \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$