On my honor, as a student,
I have neither given nor received
unauthorized aid on this academic work. **Initials:** ______

Do not write in this area

Math 116 — Final Exam — December 14, 2018

Your Initials Only: ______ Your U-M ID # (not uniqname): ____________

Instructor Name: UNOFFICIAL SOLUTIONS Section #: ________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. Do not separate the pages. If pages do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
4. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form.* Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	15	
2	9	
3	10	
4	9	
5	10	
6	6	

Problem	Points	Score
7	6	
8	8	
9	12	
10	6	
11	9	
Total	100	

1. [15 points] The table below gives several values of a twice differentiable function f along with its derivative f' and continuous second derivative f''

x	0	1	2	3	4	5	6
$f(x)$	1	2.4	2.5	2.2	2.6	4.3	6.7
$f'(x)$	2	0.7	-0.3	-0.1	1.1	2.2	2.2
$f''(x)$	-1	-1.4	-0.5	0.8	1.4	0.7	-0.7

Unless otherwise stated, you do not have to show work, but work shown might be considered for partial credit.

a. [3 points] Find the value of $\int_1^4 xf''(x)\,dx$.

Let $u = x \quad v' = f''(x)$

$u' = 1 \quad v = f'(x)$

$$\int xf''(x)dx = \int uv'dx = uv - \int u'v\,dx = xf'(x) - \int (1)f'(x)dx = xf'(x) - f(x) + C.$$

$$\text{So } \int_1^4 xf''(x)\,dx = xf'(x) - f(x)\Big|_1^4 = [4f'(4) - f(4)] - [1f'(1) - f(1)] = [4(1.1) - 2.6] - [0.7 - 2.4]$$

Answer: $\int_1^4 xf''(x)\,dx =$ 3.5

b. [3 points] Let $H(x) = \int_x^{x^2+1} f'(3t)\,dt$. Compute $H'(1)$.

$$H(x) = \frac{1}{3}f(3t)\Big|_x^{x^2+1} = \frac{1}{3}f(3x^2+3) - \frac{1}{3}f(3x)$$

$$\text{so } H'(x) = \frac{1}{3}f'(3x^2+3)(6x) - f'(3x)$$

$$\text{so } H'(1) = \frac{1}{3}f'(6)(6) - f'(3)$$

Answer: $H'(1) =$ 4.5

c. [3 points] Use TRAP(3) to approximate $\int_0^6 f(x)\,dx$. Write out each term in your sum.

$$\text{TRAP}(3) = \Delta x\left[\frac{1}{2}y_0 + y_1 + y_2 + \frac{1}{2}y_3\right] = (2)\left[\frac{1}{2}f(0) + f(2) + f(4) + \frac{1}{2}f(6)\right]$$

$$= 2\left[\frac{1}{2}(1) + 2.5 + 2.6 + \frac{1}{2}(6.7)\right]$$

Answer: $\int_0^6 f(x)\,dx \approx$ 17.9

d. [3 points] Find the 2nd degree Taylor polynomial $P_2(x)$ for $f(x)$ centered at $x = 3$.

$$P_2(x) = f(3) + f'(3)(x-3) + \frac{1}{2}f''(3)(x-3)^2$$

$$= 2.2 + (-0.1)(x-3) + \frac{1}{2}(0.8)(x-3)^2$$

Answer: $P_2(x) =$ $2.2 - 0.1(x-3) + 0.4(x-3)^2$

e. [3 points] Use your answer to part (d) to approximate $\int_0^6 f(x)\,dx$.

$$\int_0^6 f(x)dx \approx \int_0^6 P_2(x)\,dx = \int_0^6 (2.2 - 0.1(x-3) + 0.4(x-3)^2)dx$$

$$= \int_{-3}^{3} (2.2 - 0.1\omega + 0.4\omega^2)\,d\omega = 2.2\omega - 0.05\omega^2 + \frac{0.4}{3}\omega^3\Big|_{-3}^{3}$$

let $\omega = x - 3$

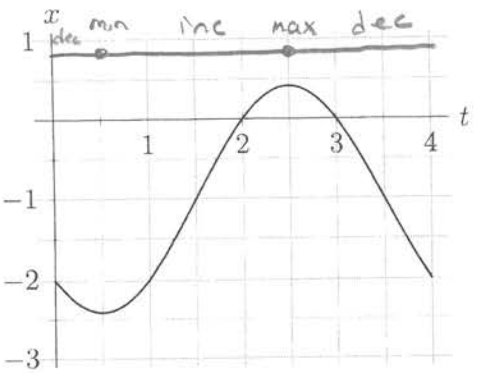
$d\omega = dx$

$x = 0 \Rightarrow \omega = -3$

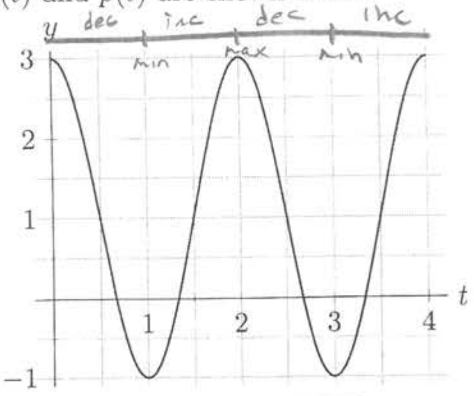
$x = 6 \Rightarrow \omega = 3$

Answer: $\int_0^6 f(x)\,dx \approx$ 20.4

2. [9 points] A particle travels along the path given by the parametric equations $x = h(t)$ and $y = p(t)$ for $0 \le t \le 4$. Graphs of the functions $h(t)$ and $p(t)$ are shown below.



Graph of $x = h(t)$



Graph of $y = p(t)$

Note: The local minima and maxima of the functions h and p are as they appear in the graphs. In particular, they occur at integer or half-integer values of t.

a. [2 points] In what interval(s) is the particle moving to the left and upwards? Circle ALL intervals below during which the particle is always moving to the left and upwards. (Circle NONE OF THESE if appropriate.)

Left: h dec
(0, ½), (2.5, 4)

up: p inc
(1, 2), (3, 4)

$0 < t < 0.5$ $0.5 < t < 1$ $1 < t < 1.5$ $1.5 < t < 2$ $2 < t < 2.5$

$2.5 < t < 3$ [✓ $3 < t < 3.5$]   [✓ $3.5 < t < 4$]   NONE OF THESE

b. [2 points] Find the equations of all horizontal tangent lines to the path of this particle for $0 < t < 4$. Write NONE if there are no horizontal tangent lines.

horizontal $\Rightarrow 0 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{p'(t)}{h'(t)} \Rightarrow p'(t) = 0.$

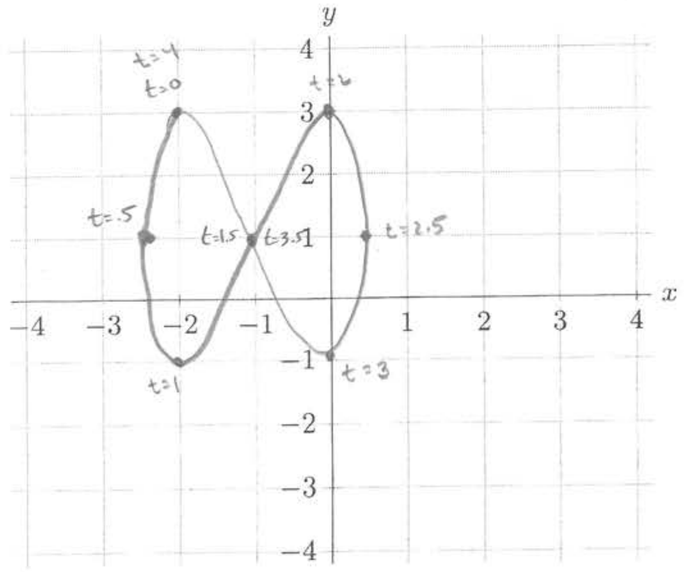
so horizontal tangents at $t = 1, 2, 3$
where $y = -1, 3, -1$

Answer: $y = -1, \quad y = 3$

c. [5 points] On the axes below, sketch a graph of the path along which the particle moves between time $t = 0$ and $t = 4$.

On your sketch, label the points described below with the corresponding value(s) of t.

- The particle's position at times $t = 0$, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, and 4.
- The points at which the particle is highest, lowest, farthest right, and farthest left.



3. [10 points]

a. [4 points] December is a busy time for cookie bakers and cookie eaters. Suppose that there is so much baking going on that cookies are added to the cookie supply of Ann Arbor at a rate of 10 pounds per minute. At the same time, 2% of the cookies are eaten every minute. Write a differential equation for the number of pounds C of cookies in Ann Arbor at time t, in minutes.

Answer: $\frac{dC}{dt} = 10 - .02C$

Let P be the number of pounds of cookies in East Lansing at time t. P satisfies the differential equation $\frac{dP}{dt} = (P-200)^2$.

b. [4 points] Suppose $P(0) = 0$. Use separation of variables to find a formula for $P(t)$. Show your work carefully.

$\int \frac{dP}{(P-200)^2} = \int dt$

let $w = P-200$

$\Rightarrow \int \frac{dw}{w^2} = \int dt$

$\Rightarrow -\frac{1}{w} = t + C$

$\Rightarrow w = \frac{-1}{t+C}$

$\Rightarrow P = 200 - \frac{1}{t+C}$

Since $P(0) = 0$,

$0 = 200 - \frac{1}{0+C}$

$\Rightarrow C = \frac{1}{200}$.

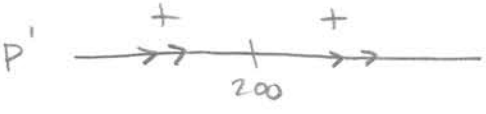
So $P(t) = 200 - \frac{1}{t+\frac{1}{200}}$

$= 200\left[1 - \frac{1}{200t+1}\right]$

$= \frac{40000t}{200t+1}$

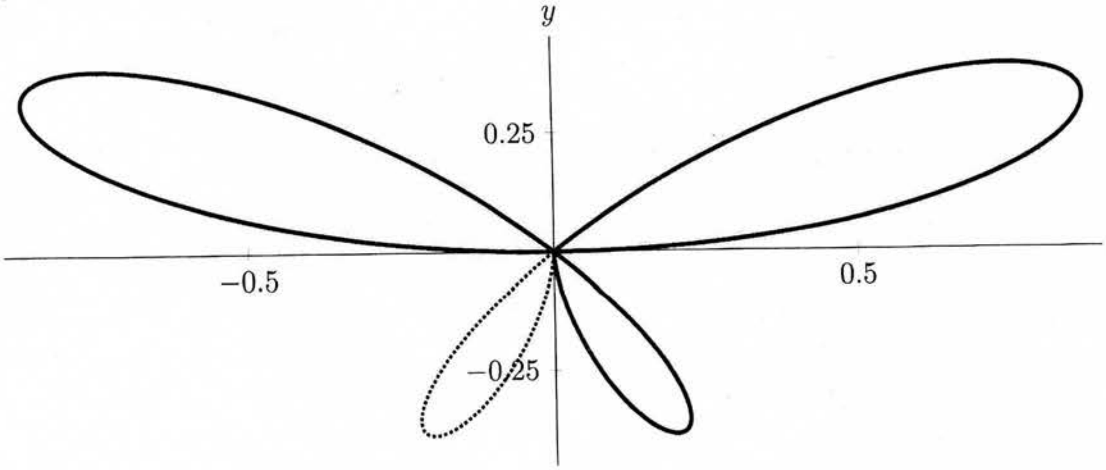
Answer: $P(t) = 200 - \frac{1}{t+\frac{1}{200}}$

c. [2 points] Find all equilibrium solutions to the differential equation $\frac{dP}{dt} = (P-200)^2$. Then determine whether or not each equilibrium solution is stable, and write each equilibrium solution on the appropriate answer blank below. Write "NONE" in the answer blank if appropriate.



Answer: Equilibrium Solution(s) Stable: NONE Not Stable: 200

4. [9 points] The polar curve $r = \sin(4\theta)\cos(\theta)$ for $0 \le \theta \le \pi$ is shown below.



Note that there are two "large loops" and two "small loops".

For reference, note that for this curve, $\dfrac{dr}{d\theta} = 4\cos(\theta)\cos(4\theta) - \sin(\theta)\sin(4\theta)$

a. [3 points] For what values of θ does the polar curve $r = \sin(4\theta)\cos(\theta)$ trace once around the "small loop" in the third quadrant? (This portion of the curve is indicated by the dotted line.) Give your answer as an interval of θ values between 0 and π.

Look at signs of x and y to determine quadrant of points:

0	$\sin\theta$	$\cos\theta$	$\sin 4\theta$	$r = \sin4\theta\cos\theta$	$x = r\cos\theta$	$y = r\sin\theta$	
	+	+	+	+	+	+	Q1
$\pi/4$							
	+	+	−	−	−	−	$\boxed{\text{Q3}}$
$\pi/2$							
	+	−	+	−	+	−	Q4
$3\pi/4$							
	+	−	−	+	−	+	Q2
π							

Answer: $\boxed{\frac{\pi}{4} < \theta < \frac{\pi}{2}}$

b. [3 points] Write, but do not evaluate, an expression involving one or more integrals that gives the total arc length of the two small loops.

$$\text{Arc len} = 2\int_{\pi/4}^{\pi/2} \sqrt{r^2 + (r')^2}\, d\theta$$

Answer: Arc Length = $2\int_{\pi/4}^{\pi/2} \sqrt{(\sin4\theta\cos\theta)^2 + (4\cos4\theta\cos\theta - \sin4\theta\sin\theta)^2}\, d\theta$

c. [3 points] Write, but do not evaluate, an expression involving one or more integrals that gives the area of the region that is enclosed by the polar curve $r = 2$ but is outside the curve $r = \sin(4\theta)\cos(\theta)$.

$|r| = |\sin4\theta| \cdot |\cos\theta| \le 1 \cdot 1 < 2$, So the butterfly is contained in the circle of radius 2.

Area inside butterfly $= \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int_0^{\pi} \sin^2 4\theta \cos^2\theta\, d\theta$

Answer: Area = $\boxed{4\pi - \frac{1}{2}\int_0^{\pi} \sin^2 4\theta \cos^2\theta\, d\theta}$

5. [10 points]

a. [5 points] Using an appropriate Taylor series for e^x, determine whether the integral $\displaystyle\int_0^1 \frac{1}{e^{\sqrt{x}}-1}\,dx$ converges.

Circle one: [✓ CONVERGES] DIVERGES

Justification:

$$e^{\sqrt{x}} = 1 + \sqrt{x} + \frac{1}{2!}(\sqrt{x})^2 + \frac{1}{3!}(\sqrt{x})^3 + \cdots \geq 1+\sqrt{x} \quad \text{if } 0<x\leq 1.$$

So $e^{\sqrt{x}} - 1 \geq (1+\sqrt{x}) - 1 = \sqrt{x}$

So $\dfrac{1}{e^{\sqrt{x}}-1} \leq \dfrac{1}{\sqrt{x}}$

$\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 \frac{1}{x^{1/2}}\,dx$ converges by the p-test $(p = 1/2)$

So $\displaystyle\int_0^1 \frac{1}{e^{\sqrt{x}}-1}\,dx$ converges by comparison.

b. [5 points] Using a Taylor series for a function $f(x)$, compute the exact value of $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^{n-1}}$.

(Hint: Consider the function $\dfrac{1}{1-x}$.)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

Take derivatives of both sides:

$-(1-x)^{-2}(-1) = \displaystyle\sum_{n=0}^{\infty} n x^{n-1}$. The $n=0$ term is 0 so can ignore it:

$$\Rightarrow \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

That converges by the ratio test when $|x|<1$. So if $x = \frac{1}{3}$,

$$\sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^{n-1} = \frac{1}{(1-\frac{1}{3})^2}$$

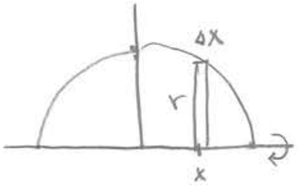
Answer: This is a value of a Taylor series for the function $f(x) = \dfrac{1}{(1-x)^2}$

The radius of convergence of this Taylor series is 1 (No justification needed)

Finally, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^{n-1}} = 9/4 = 2.25$

6. [6 points] Consider the curve $y = \sqrt{1-x^2}$. Suppose a paperweight is formed by rotating this curve around the x-axis. This paperweight has a density given by $\rho(x) = 2 + \cos(x)$ g/cm^3. The units on both axes are centimeters (cm).

a. [3 points] Write an expression that gives the approximate mass, in grams, of a slice of the paperweight taken perpendicular to the x-axis at coordinate x with thickness Δx. (Assume that Δx is small but positive.) Your expression should <u>not</u> involve any integrals.



radius of slice = $\sqrt{1-x^2}$ cm

volume of slice = $\pi r^2 \Delta x = \pi(1-x^2)\Delta x$ cm^3

mass of slice = $\rho(x) \cdot$ vol

Answer: Mass of slice ≈ $(2+\cos x)\,\pi(1-x^2)\,\Delta x$

b. [3 points] Write, but do <u>not</u> evaluate, an expression involving one or more integrals that gives the mass, in grams, of the paperweight.

Answer: Mass = $\int_{-1}^{1} (2+\cos x)\,\pi(1-x^2)\,dx$

7. [6 points] Determine whether the following series converges absolutely, converges conditionally, or diverges, and give a complete argument justifying your answer. In particular, be sure to show all work and include any convergence tests used.

Correction made at time of exam

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$$

Circle one: CONVERGES ABSOLUTELY [✓ CONVERGES CONDITIONALLY] DIVERGES

Justification:

- terms alternate in sign
- |terms| decreases
- |terms| $\to 0$.

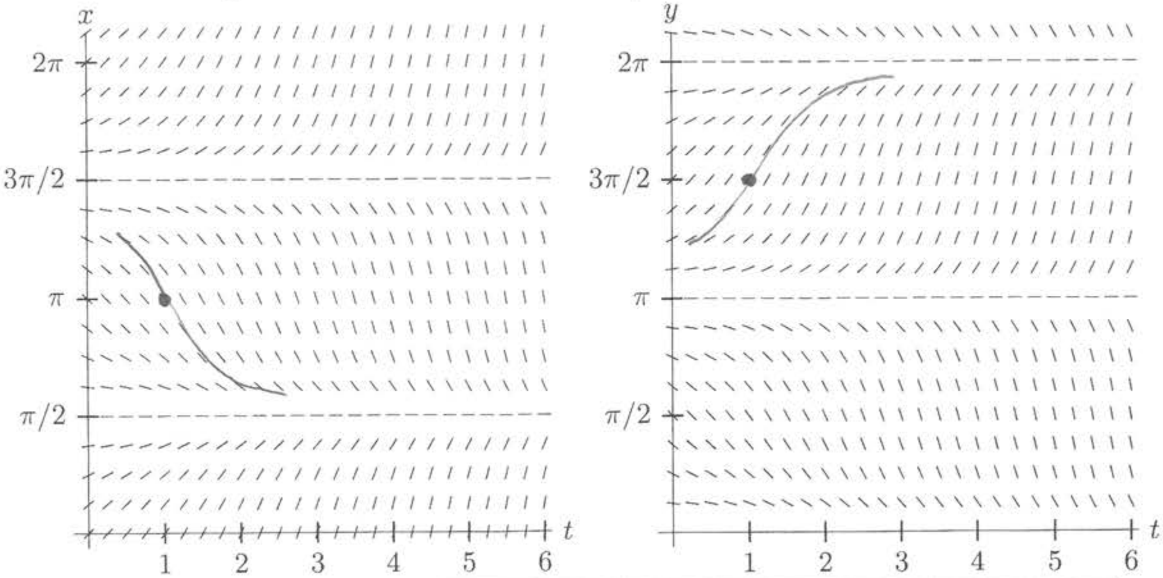
So converges by the alternating series test.

<u>But</u> $\frac{\ln(n)}{n} \geq \frac{1}{n}$ eventually, and $\sum \frac{1}{n}$ diverges by the p-test ($p=1$). So $\sum \frac{\ln(n)}{n}$ diverges by comparison.

8. [8 points] Suppose Xena the cat is running around the backyard while wearing a tracking device on her collar. The device measures her x- and y-coordinates in meters, with the origin set as the center of the yard. Suppose t is measured in minutes after Xena went outside. Xena's x- and y- coordinates satisfy the following differential equations for $t > 0$:

$$\frac{dx}{dt} = \cos(x)\sqrt{t^2 + \cos^2(x)} \quad \text{and} \quad \frac{dy}{dt} = -\sin(y)\sqrt{t^2 + \sin^2(y)}$$

Portions of the slope fields for these differential equations are shown below.



Note the following: One minute after she goes outside, Xena is at the point $(\pi, 3\pi/2)$.

a. [3 points] How fast is Xena traveling one minute after she goes outside?

So $x = \pi$, $y = \frac{3\pi}{2}$

$$\left.\frac{dx}{dt}\right|_{t=1} = \cos(\pi)\sqrt{1^2 + \cos^2\pi} = (-1)\sqrt{1^2+(-1)^2} = -\sqrt{2}$$

$$\left.\frac{dy}{dt}\right|_{t=1} = -\sin\left(\frac{3\pi}{2}\right)\sqrt{1^2 + \sin^2\left(\frac{3\pi}{2}\right)} = -(-1)\sqrt{1^2+(-1)^2} = \sqrt{2}$$

$$\text{Speed} = \sqrt{x'^2 + y'^2} = \sqrt{2+2} = 2$$

Answer: Speed = 2 meters per minute

b. [3 points] Find the equation (in xy-coordinates) of the line tangent to Xena's path one minute after she goes outside.

$$\frac{dy}{dx} = \frac{dx/dt}{dy/dt}, \text{ so at } t=1 \ \frac{dy}{dx} = \frac{\sqrt{2}}{-\sqrt{2}} = -1$$

Answer: $y = (-1)(x-\pi) + \frac{3\pi}{2} = \frac{5\pi}{2} - x$

c. [2 points] If Xena keeps running around the yard for a long time, what point (in xy-coordinates) in the yard will she approach?

Answer: $\left(\frac{\pi}{2}, 2\pi\right)$

9. [12 points] For each of the questions on this page:
You must circle at least one choice to receive any credit.
No credit will be awarded for unclear markings. No justification is necessary.

For parts **a** - **c** below, circle <u>all</u> of the available correct answers, and circle "NONE OF THESE" if none of the available options are correct.

a. [4 points] Suppose a_n and b_n are nonzero sequences. Functions P and Q satisfy the following: $P(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$ for $-1 < x \leq 3$ and $Q(x) = \sum_{n=0}^{\infty} b_n x^n$ for $-1 \leq x \leq 1$. Which of the following <u>must</u> be true?

[✓ i. The radius of convergence of the Taylor series for $P(x)$ around $x = 1$ is at least 1.]

$\left|\frac{b_n}{n}\right| < |b_n|$

[✓ ii. $\sum_{n=1}^{\infty} \frac{b_n}{n}$ converges.]

iii. $\sum_{n=0}^{\infty} a_n 2^n$ diverges.

[✓ iv. $\sum_{n=0}^{\infty} \frac{1}{a_n}$ diverges.]

Since $P(2) = \sum a_n$ converges, $a_n \to 0$ so $\frac{1}{a_n} \to \infty$

v. The Taylor series for $P(x)$ around $x = 0$ is $\sum_{n=0}^{\infty} a_n x^n$.

vi. NONE OF THESE

b. [4 points] Suppose $f(x)$ is a positive, decreasing, and concave up function. Suppose further that all derivatives of $f(x)$ exist at $x = 0$. Define $F(x) = \int_0^x f(t)\,dt$. Which of the following <u>must</u> be true?

f pos, dec $\Rightarrow$ F inc, conc Down

f dec, conc up $\Rightarrow$ f' neg, inc

False $\rightarrow$ i. TRAP(n) is an overestimate of $\int_0^1 F(x)\,dx$ for all positive integers n.

since F conc Dn

F, f' both inc

[✓ ii. $F(x) + F''(x)$ is an increasing function.]

if $F = \sum a_n x^n$, $f = \sum n a_n x^{n-1}$

Ratio test: $\sum a_n x^n$ conv if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x| < 1$

$\sum n a_n x^{n-1}$ conv if $\lim_{n\to\infty} \left|\frac{(n+1)a_{n+1}}{n a_n}\right| |x| < 1$, but those limits are the same.

[✓ iii. The Taylor series for $F(x)$ and for $f(x)$ centered around $x = 0$ both have the same radius of convergence.]

$\int_0^1 \frac{f(x)}{F(x)}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{f(x)}{F(x)}\,dx$

let $w = F(x) \Rightarrow dw = f(x)\,dx$

$= \lim_{a\to 0^+} \int_{F(a)}^{F(1)} \frac{dw}{w}$

$= \int_0^{F(1)} \frac{dw}{w}$

which diverges by p-test ($p=1$)

iv. $\int_0^1 \frac{f(x)}{F(x)}\,dx$ converges.

v. $\sum_{n=1}^{\infty} f(n)$ converges.

↑ counterexample: $f(x) = 1 + e^{-x}$

vi. NONE OF THESE

c. [4 points] Consider the differential equation $y' = (\cos(x) - \sin(y))^2$, and suppose $y = g(x)$ is the solution to this differential equation that passes through the point $(0, 0)$. Which of the following <u>must</u> be true?

[✓ i. This differential equation has no equilibrium solutions.]

[✓ ii. $g''(0) = -2$.]

iii. $y = \arcsin(\cos(x))$ is an equilibrium solution.

[✓ iv. $g(x) \leq 4x$ for all $x > 0$]

because $y' \leq 4$

[✓ v. $g(x)$ is increasing.]

because $y' \geq 0$

vi. NONE OF THESE

10. [6 points] The Taylor series centered at $x = 0$ for a function $F(x)$ converges to $F(x)$ for $-e^{-1} < x < e^{-1}$ and is given below.

$$F(x) = \sum_{n=0}^{\infty} \frac{(n+1)^n}{n!} x^n \quad \text{for } -\frac{1}{e} < x < \frac{1}{e}.$$

a. [2 points] What is $F^{(2018)}(0)$? Make sure your answer is exact. You do not need to simplify.

$$\frac{F^{(2018)}(0)}{2018!} = \text{coefficient of } x^{2018} = \frac{(2018+1)^{2018}}{(2018)!}$$

Answer: $F^{(2018)}(0) =$ 2019^{2018}

b. [4 points] Use appropriate Taylor series for $F(x)$ and $\cos(x)$ to compute the following limit:

$$\lim_{x \to 0} \frac{(F(x)-1)(\cos(x)-1)}{x^3}$$

Show your work carefully.

$$F(x) = \frac{(0+1)^0}{0!} x^0 + \frac{(1+1)^1}{1!} x^1 + \frac{(2+1)^2}{2!} x^2 + \cdots$$

$$= 1 + 2x + \frac{9}{2} x^2 + \cdots$$

so $F(x) - 1 = 2x + \frac{9}{2} x^2 + \cdots$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

so $\cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{24} - \cdots$

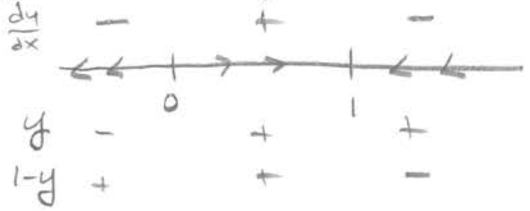
so we have $\displaystyle \lim_{x \to 0} \frac{(2x + \frac{9}{2}x^2 + \cdots)(-\frac{x^2}{2} + \frac{x^4}{24} - \cdots)}{x^3} = \lim_{x \to 0} \frac{-x^3 + (\text{powers of } x \text{ bigger than } 3)}{x^3}$

Answer: $\displaystyle \lim_{x \to 0} \frac{(F(x)-1)(\cos(x)-1)}{x^3} =$ -1

11. [9 points] Provide an example for each of the following. Your example must clearly satisfy the given properties. If no example exists then write DOES NOT EXIST and briefly explain why no such example exists.

a. [3 points] A differential equation that has at least one equilibrium solution that is stable and at least one equilibrium solution that is not stable. (A complete answer consists of the differential equation and both of these equilibrium solutions.)

$$\frac{dy}{dx} = y(1-y)$$

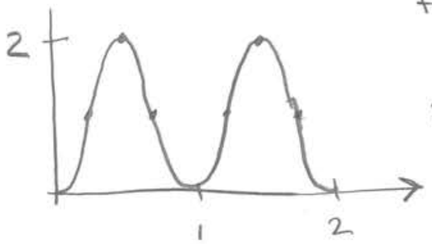


Answer: Differential Equation: dy/dx = $y(1-y)$

Equilibrium Solutions: Stable: 1 Not Stable: 0

b. [3 points] A continuous function $f(x)$ such that $\text{LEFT}(2) \le \text{RIGHT}(2) \le \int_0^2 f(x)\,dx$ where LEFT(2) and RIGHT(2) are, respectively, the left- and right- hand Riemann sum estimates for $\int_0^2 f(x)\,dx$ with two equal subintervals.

You may describe your function f by giving a formula or by drawing a clear and well-labeled graph. Then briefly explain why your function is indeed such an example.



$f(x) = 1 - \cos(2\pi x)$

$\text{LEFT}(2) = (1)\left[f(0) + f(1)\right] = 0$

$\text{RIGHT}(2) = (1)\left[f(1) + f(2)\right] = 0$

But $\int_0^2 f(x)\,dx = x - \frac{1}{2\pi}\sin(2\pi x)\Big|_0^2 = [2-0] - [0-0] = 2$

Answer: $f(x) =$ $1-\cos(2\pi x)$

Brief explanation:

If f is neither increasing nor decreasing, it's possible for both LEFT and RIGHT to be over or underestimates. In this case, the sample points happen to be the lowest values of the function.

c. [3 points] Give an example of a power series whose radius of convergence is 0. (You must use sigma notation for full credit.)

How about: $\sum_{n=0}^{\infty} n!\,x^n$, By the ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|$

$= \lim_{n\to\infty}(n+1)|x|$ which diverges unless $x = 0$.

Answer: Power Series: $\sum_{n=0}^{\infty} n!\,x^n$

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Normal Distributions

The density function of a normal distribution with mean μ and standard deviation $\sigma > 0$ is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

The standard normal distribution is the normal distribution with $\mu = 0$ and $\sigma = 1$.