Math 116 — First Midterm — October 7, 2019

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. Do not separate the pages of this exam. If pages do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
4. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	14	
3	9	
4	8	
5	10	

Problem	Points	Score
6	7	
7	9	
8	11	
9	12	
10	8	
Total	100	

1. [12 points] Let $h(x)$ be a differentiable function with continuous and differentiable derivative. Some values of the functions $h(x)$ and $h'(x)$ are shown below:

x	1	2	3	4	7	8	9
$h(x)$	-2	5	-0.1	0.5	0	-1	4
$h'(x)$	8	3	1	2	8	-1	-4

Compute the exact numerical values of the following integrals. If it is not possible to do so based on the information provided, write "NOT POSSIBLE" and clearly indicate why it is not possible. Show your work.

a. [4 points] $\displaystyle\int_2^3 h'(h'(x))h''(x)\,dx$

Solution:
We use substitution with $w = h'(x)$.

$$\begin{aligned}\int_2^3 h'(h'(x))h''(x)\,dx &= \int_3^1 h'(w)dw \\ &= h(w)\Big|_3^1 \\ &= h(1) - h(3) = -2 - (-0.1) = -1.9\end{aligned}$$

Answer: -1.9

b. [4 points] $\displaystyle\int_1^2 (1-x)h''(x)\,dx$

Solution: We use integration by parts.

$$\begin{aligned}\int_1^2 (1-x)h''(x)\,dx &= (1-x)h'(x)\Big|_1^2 + \int_1^2 h'(x)\,dx \\ &= (1-x)h'(x)\Big|_1^2 + h(x)\Big|_1^2 \\ &= -h'(2) - 0 + h(2) - h(1) = -3 + 5 + 2 = 4\end{aligned}$$

Answer: 4

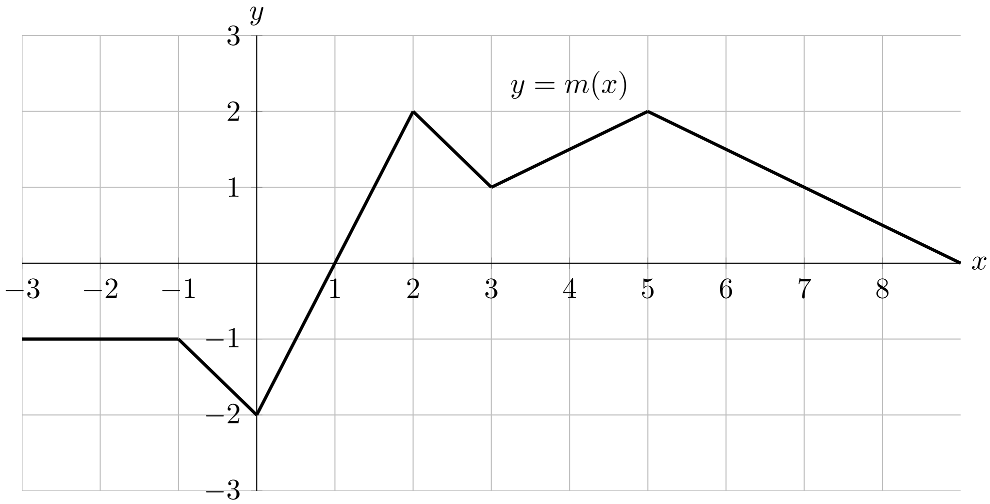
c. [4 points] $\displaystyle\int_4^9 \frac{h'(\sqrt{x})}{\sqrt{x}}\,dx$

Solution: We use substitution with $w = \sqrt{x}$.

$$\begin{aligned}\int_4^9 \frac{h'(\sqrt{x})}{\sqrt{x}}\,dx &= 2\int_2^3 h'(w)\,dw \\ &= 2h(w)\Big|_2^3 = 2(-0.1 - 5) = -10.2\end{aligned}$$

Answer: -10.2

2. [14 points] Part of the graph of a continuous, piecewise-linear function $m(x)$ is given below. The domain of $m(x)$ is all real numbers.



Let:

- $F(x) = \int_1^x m(t)\,dt$
- $G(x) = \int_2^{x/2} m(t)\,dt$
- $H(x)$ is an antiderivative of $m(x)$ with $H(2) = 8$.

You do not need to show work for this problem.

a. [11 points] Find the following values. If it is not possible to do so based on the information provided, write "NI". If the value does not exist, write "DNE".

(i) $F(1) =$ 0

(ii) $F(3) =$ 2.5

(iii) $F(-2) =$ 3.5

(iv) $F'(4) =$ 1.5

(v) $G(2) =$ -1

(vi) $G(6) =$ 1.5

(vii) $G'(8) =$ 0.75

(viii) $H(3) =$ 9.5

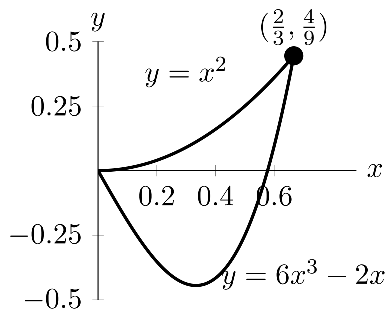
(ix) $H(10) - F(10) =$ 7

b. [3 points] On which of the following intervals is $H(x)$ concave up on the entire given interval? Circle all correct answers.

$\boxed{(0, 2)}$ $(1, 3)$ $(2, 5)$ $\boxed{(3, 5)}$ NONE OF THESE

3. [9 points] Scientists are studying a cave. The inside of the cave can be modeled as a solid in the following way:

- the base is a region bounded by the curves $y = 6x^3 - 2x$ and $y = x^2$
- cross-sections perpendicular to the $x-$axis are squares
- the cave's entrance is at the origin
- x is measured in miles east of the entrance, and y is measured in miles north of the entrance



a. [6 points] The scientists want to know the volume of the cave.

(i) Write, but **do not evaluate**, an expression that gives the approximate volume, in cubic miles, of a vertical slice of the cave that is Δx miles thick and x miles east from the entrance of the cave.

Answer: Volume of slice ≈ $(x^2 - (6x^3 - 2x))^2 \Delta x$

(ii) Write, but do not evaluate, an expression involving one or more integrals that gives the total volume, in cubic miles, of the cave.

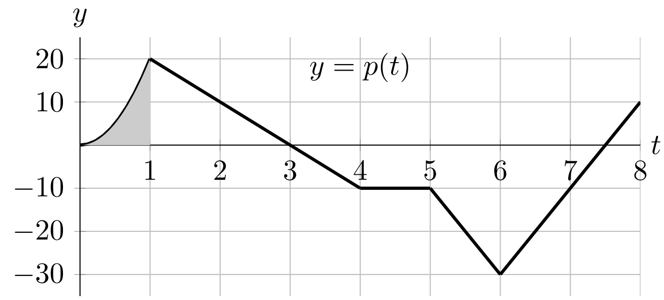
Answer: $\int_0^{2/3} (x^2 - (6x^3 - 2x))^2 dx$

b. [3 points] Salamanders live on the floor of the cave at a density that depends on the x distance from the entrance. Let $\delta(x)$ be their population density, in salamanders per square mile. Write, but do not evaluate, an expression involving one or more integrals that gives the total number of salamanders living in the cave.

Answer: $\int_0^{2/3} (x^2 - (6x^3 - 2x))\delta(x) dx$

4. [8 points] In the cartoon adaptation of *Derivative Girl*, Darth Integrator dumps toxins (which will make everyone forget their calculus knowledge) into a local park's lake.

Derivative Girl discovers this October 10 at midnight (12am, 00:00), and tries her best to clear out the toxin in the lake. The rate at which the total volume of toxin in the lake changes, in gallons per day, is given by $p(t)$, where t is measured in days since October 10. The graph of $p(t)$ is shown below.



On October 15 at midnight, Derivative Girl realized there were still 50 gallons of toxin in the lake. Note that the shaded region in the graph above has area 7.

Let $P(t)$ be the total number of gallons of toxin in the lake t days after October 10 at midnight.

a. [2 points] How many days after Derivative Girl discovers the toxin is the volume of toxin in the lake at its largest?

Answer: **3**

b. [2 points] How many days after Derivative Girl discovers the toxin is the amount of toxin in the lake growing fastest?

Answer: **1**

c. [2 points] Write an expression involving one or more integrals that gives $P(t)$ in terms of the function p.

Answer: $P(t) = \int_5^t p(x)\,dx + 50$

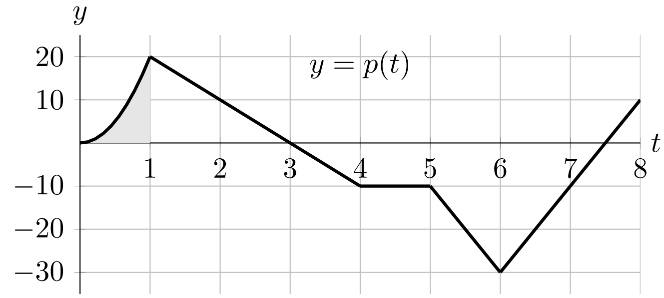
d. [2 points] In the context of this problem, give a practical interpretation of the expression

$$\frac{1}{5-2}\int_2^5 P(t)\,dt.$$

Include units.

Solution: The average volume of toxin in the lake from October 12 to October 15 is $\dfrac{1}{5-2}\displaystyle\int_2^5 P(t)\,dt$ gallons.

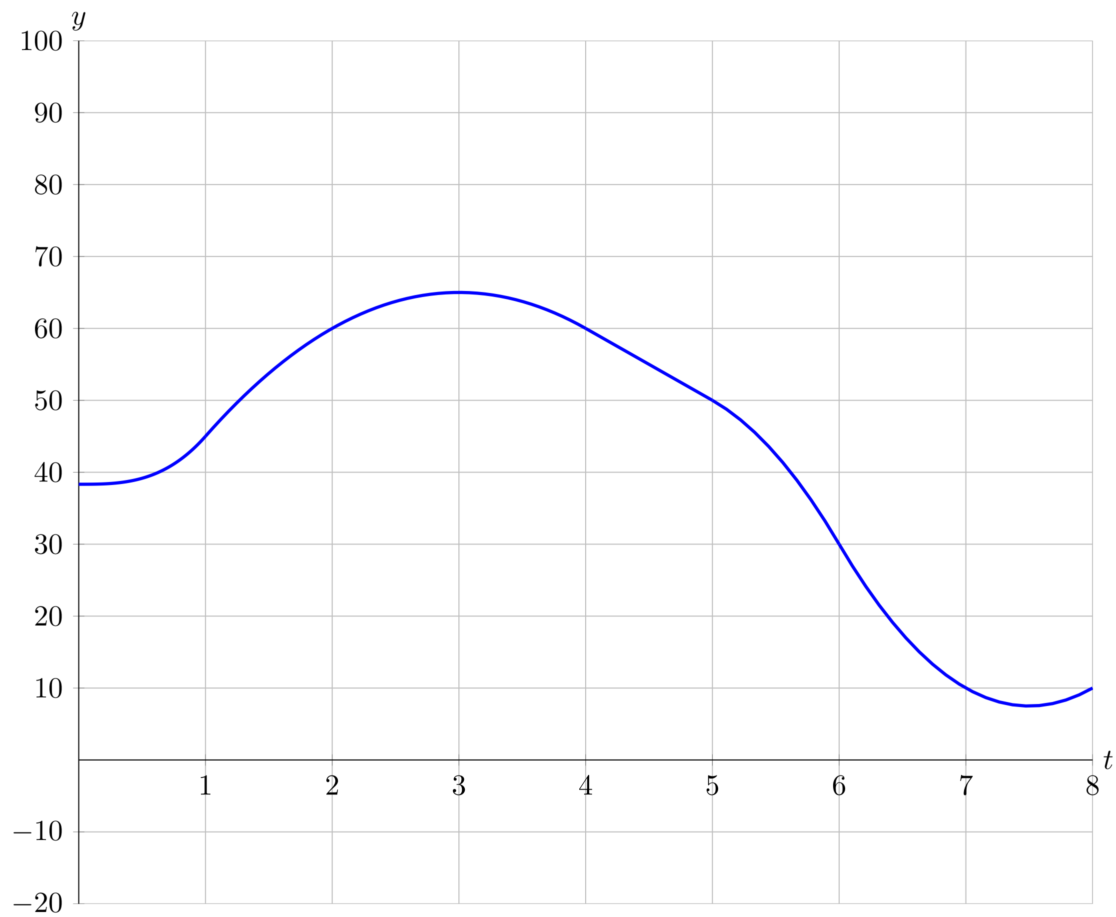
5. [10 points] This problem uses the same setup as the previous problem. Recall that the rate at which the total volume of toxin in the lake changes, in gallons per day, is given by $p(t)$, where t is measured in days after midnight on October 10. The graph of $p(t)$ is shown below.



Recall that the area of the shaded region is 7, and that on October 15 at midnight, Derivative Girl realized there were still 50 gallons of toxin in the lake.

Let $P(t)$ be the total number of gallons of toxin in the lake t days after October 10 at midnight. Carefully sketch the graph of $P(t)$ below. Make sure the following features of the function $P(t)$ are displayed clearly in your sketch:

- the value of P at $t = 0, 1, 2, \ldots, 8$.
- where $P(t)$ is and is not differentiable;
- where $P(t)$ is increasing, decreasing, or constant;
- the concavity of $P(t)$.



6. [7 points] Does the following integral converge or diverge? Be sure to show all work and indicate any theorems you use.

$$\int_{10}^{\infty} \frac{5x+\cos(x)-1}{2x^3+2x+7}dx$$

Answer *(Circle one):* **Diverges** $\boxed{\textbf{Converges}}$

Justification:

Solution: Since $\cos(x) \leq 1$, we have $5x + \cos(x) - 1 \leq 5x$.
We also know $2x^3 + 2x + 7 \geq 2x^3$ for all $x \geq 0$. Therefore

$$\frac{5x+\cos(x)-1}{2x^3+2x+7} < \frac{5x}{2x^3} = \frac{5}{2x^2}$$

for $x \geq 10$.
By the p-test, with $p = 2 > 1$, we know that $\int_{10}^{\infty} \frac{5}{2x^2}\,dx$ converges.
Therefore, by the (Direct) Comparison Test for improper integrals, $\int_{10}^{\infty} \frac{5x+\cos(x)-1}{2x^3+2x+7}dx$ also converges.

7. [9 points] For each of the questions below, circle **all** of the available correct answers. Circle "NONE OF THESE" if none of the available choices are correct. You must circle at least one choice to receive any credit. No credit will be awarded for unclear markings. No justification is necessary

a. [3 points] Let $h(z)$ be an even differentiable function. Which of the following expressions must be equal to $\displaystyle\int_{-\pi/2}^{\pi/2} \cos(z)\, h(\sin(z))\, dz$?

$\boxed{\text{i. } \displaystyle\int_{-1}^{1} h(w)\, dw}$

$\boxed{\text{ii. } 2\displaystyle\int_{0}^{1} h(w)\, dw}$

iii. $\displaystyle\int_{-\pi/2}^{\pi/2} h(w)\, dw$

iv. $\displaystyle\int_{0}^{z} \cos(t)\, h(\sin(t))\, dt$

v. $\displaystyle\int_{-\pi/2}^{\pi/2} \cos(z)\, dz \cdot \int_{-\pi/2}^{\pi/2} h(\sin(z))\, dz$

$\boxed{\text{vi. } \sin(z)h(\sin(z))\Big|_{-\pi/2}^{\pi/2} - \displaystyle\int_{-\pi/2}^{\pi/2} \sin(z)\cos(z)h'(\sin(z))\, dz}$

vii. NONE OF THESE

b. [3 points] Which of the following is an antiderivative of $e^{\cos(x)}$ passing through $(2, 3)$?

i. $\displaystyle\int_{0}^{2} e^{\cos(t)}\, dt + 3$

$\boxed{\text{ii. } \displaystyle\int_{0}^{x-2} e^{\cos(t+2)}\, dt + 3}$

$\boxed{\text{iii. } \displaystyle\int_{2}^{x} e^{\cos(t)}\, dt + 3}$

iv. $\displaystyle\int_{2}^{x} e^{\cos(t)} + 3\, dt$

v. $-\sin(x)e^{\cos(x)}+3$

vi. $-\dfrac{1}{\sin(x)}e^{\cos(x)}+3$

vii. $-\dfrac{1}{\sin(x)}e^{\cos(x)} + \dfrac{1}{\sin(2)}e^{\cos(2)} + 3$

viii. NONE OF THESE

c. [3 points] Consider the integral $\displaystyle\int_{2}^{\infty} \frac{1}{e^{-x}+x^2}\, dx$. Which of the following statements are true?

i. We can use the comparison test with $\dfrac{1}{e^{-x}}$ to conclude that the integral converges.

ii. We can use the comparison test with $\dfrac{1}{e^{-x}}$ to conclude that the integral diverges.

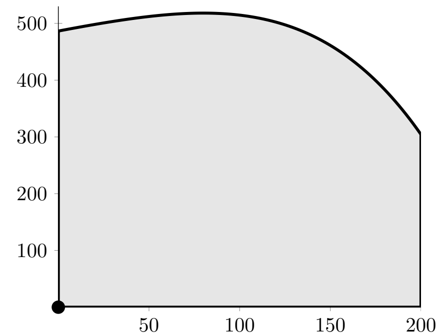
iii. $\boxed{\text{We can use the comparison test with } \dfrac{1}{x^2} \text{ to conclude that the integral converges.}}$

iv. We can use the comparison test with $\dfrac{1}{x^2}$ to conclude that the integral diverges.

v. NONE OF THESE

8. [11 points]

A city is designing a park, which will have three straight sides and one curved side, as shown in the diagram at right. There is a fountain at the southwest corner of the park, located at $(0,0)$. Let $p(x)$ be the north-south distance, in feet, across the park x feet to the east of the fountain. Note that the park is 200 feet wide in the east-west direction.



City planners have the following data for $p(x)$.

x	0	50	100	150	200
$p(x)$	487	512	515	462	305
$p'(x)$	0.606	0.364	-0.373	-1.91	-4.57

a. [6 points] The planners would like to be able to say the following:

The area of the park is at most ________ square feet.

Given this, should they use an overestimate or an underestimate to approximate the area of the park?

Answer *(circle one)*: $\boxed{\text{OVERESTIMATE}}$ UNDERESTIMATE

Which one of the following methods of approximation should they use to guarantee this?

RIGHT(n) LEFT(n) $\boxed{\text{MID}(n)}$ TRAP(n)

Find the approximation you chose above, using the maximal amount of equal subintervals possible, for the area of the park in square feet. Write out all the terms in your sum.

Answer: $\text{MID}(2) = 100(p(50) + p(150)) = 100(512 + 462) = 97400$

b. [5 points] They plan to put a fence along the entire northern (curved) side. Write an expression involving one or more integrals that gives the total length of this fence in feet.

Answer: $\int_0^{200} \sqrt{1 + (p'(x))^2}\, dx$

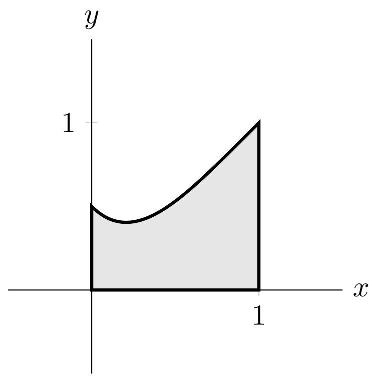
Use a RIGHT(2) approximation to estimate the length of the fence. Write out all the terms in your sum.

Answer: $100(\sqrt{1 + (-0.373)^2} + \sqrt{1 + (-4.57)^2}) = 574.543$

9. [12 points] Kyle wants to make a big ring, made by the rotation of the region bounded by

$$y = x + \frac{1}{2}(x-1)^4, \; x = 0, \; x = 1, \text{ and } y = 0$$

about the line $x = -\frac{1}{2}$. This region is shown below. Both x and y are measured in centimeters.



a. [4 points] Write, but do not evaluate, an integral expression that gives the volume of Kyle's ring in cm^3.

Solution: It would be very difficult to slice with respect to y, since we would have to split the upper part into pieces. So we will slice with respect to x, which means we must use shell method.

Answer: $\displaystyle\int_0^1 2\pi\left(x+\frac{1}{2}\right)\left(x+\frac{1}{2}(x-1)^4\right)dx$

b. [4 points] The ring's density is given by $\ln(5r+1)$ grams/cm^3, where r is the distance in centimeters from the central axis of the ring. Write, but do not evaluate, an integral expressing the total mass of Kyle's ring in grams.

Solution: We must either write the density function in terms of the x-coordinate, or the integrand above in terms of r. Since the region is being rotated around $x = -\frac{1}{2}$, we get $r = x - -\frac{1}{2} = x + \frac{1}{2}$.

Answer: $\displaystyle\int_0^1 2\pi\left(x+\frac{1}{2}\right)\left(x+\frac{1}{2}(x-1)^4\right)\left(\ln\left(5(x+\frac{1}{2})+1\right)\right)dx$

c. [4 points] John wants to use the same region to make a ring, but instead rotates the region around the line $y = -\frac{1}{2}$.
Write, but do not evaluate, an integral that gives the **volume** of John's ring in cm^3.

Answer: $\displaystyle\int_0^1 \pi\left(\left(x+\frac{1}{2}(x-1)^4+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right)dx$

10. [8 points] Two functions, $f(x)$ and $g(x)$ are continuous and differentiable for all $x > 2$, and:

- $\lim_{x\to 2+} f(x) = \infty$ (this means that $f(x)$ has a vertical asymptote at $x = 2$),
- $\frac{d}{dx}\left(\frac{3 - 3\cos(\pi x)}{g(x)}\right) = f(x)$ for all $x > 2$,
- $g(3) = 4$,
- $\lim_{x\to 2+} g(x) = 0$, and
- $\lim_{x\to 2+} g'(x) = 10$.

Determine whether the following integral converges or diverges, and if the integral converges, give its exact value. Be sure to show all work and indicate any theorems you use.

$$\int_2^3 f(x)dx$$

Answer *(Circle one):* **Diverges** $\boxed{\textbf{Converges}}$ to: **3/2**

Justification:

Solution:
The first bullet point tells us that this integral is improper, so we must start by changing to limit notation.

$$\begin{aligned}\int_2^3 f(x)dx &= \lim_{b\to 2^+}\int_b^3 f(x)dx\\ &= \lim_{b\to 2^+}\left(\frac{3-3\cos(\pi 3)}{g(3)} - \frac{3-3\cos(\pi b)}{g(b)}\right)\\ &= \lim_{b\to 2^+}\left(\frac{6}{4} - \frac{3-3\cos(\pi b)}{g(b)}\right)\\ &= \frac{3}{2} - \lim_{b\to 2^+}\frac{3-3\cos(\pi b)}{g(b)}\end{aligned}$$

At this point, we see that $\lim_{b\to 2^+} g(b) = 0$ and $\lim_{b\to 2^+} 3 - 3\cos(\pi b) = 0$, so we will apply L'Hopital's Rule to get

$$\begin{aligned}\frac{3}{2} - \lim_{b\to 2^+}\frac{3-3\cos(\pi b)}{g(b)} &= \frac{3}{2} - \lim_{b\to 2^+}\frac{3\sin(\pi b)}{g'(b)} \text{ by L'Hopital's Rule}\\ &= \frac{3}{2} - \frac{0}{10}\\ &= \frac{3}{2}\end{aligned}$$