Math 116 — Second Midterm — November 11, 2019

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. Do not separate the pages of this exam. If pages do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
4. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	9	
3	8	
4	8	
5	9	

Problem	Points	Score
6	8	
7	6	
8	8	
9	9	
10	9	
11	15	
Total	100	

1. [11 points] Parts **a.** and **b.** are unrelated. You do not need to justify your answers.

a. [8 points] Consider the following sequences, defined for all $n \geq 1$.

$$a_n = 3(0.999)^n$$

$$b_n = \sum_{k=1}^{n} 3(0.999)^k$$

$$c_n = \int_1^n 3 - \frac{1}{x}\,dx$$

$$d_n = \cos(\pi n)$$

For each of the following, circle all sequences that satisfy the given property.

(i) Which sequences are bounded?

$\boxed{a_n}$ $\boxed{b_n}$ c_n $\boxed{d_n}$

(ii) Which sequences are increasing?

a_n $\boxed{b_n}$ $\boxed{c_n}$ d_n

(iii) Which sequences are decreasing?

$\boxed{a_n}$ b_n c_n d_n

(iv) Which sequences converge?

$\boxed{a_n}$ $\boxed{b_n}$ c_n d_n

b. [3 points] Write out the first 3 terms of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(5x)^{2n}}{n^p}$, where p is a positive constant.

Answer: $(5x)^2, \dfrac{(5x)^4}{2^p}, \dfrac{(5x)^6}{3^p}$

2. [9 points] Note: *"Closed form" here means that the expression should* NOT *include sigma notation or ellipses (...) and should* NOT *be recursive.*

In the live-action series adaptation of *Derivative Girl*, Derivative Girl can lift $D_0 = 1000$ kg, and she can make as many copies of herself as she wants. The first copy can lift 1/3 the amount Derivative Girl can lift, and the nth copy can lift 1/3 the amount the $(n-1)$st copy can lift.

a. [3 points] Let D_n be the amount of mass, in kg, that the nth copy of Derivative Girl can lift. Calculate D_1 and D_2, and give a closed-form expression for D_n in terms of n:

$D_1 =$ $1000(\frac{1}{3}) \approx 333.33$

$D_2 =$ $1000(\frac{1}{3^2}) \approx 111.11$

$D_n =$ $1000(\frac{1}{3^n})$

b. [4 points] Let G_n be the amount of mass, in kg, that Derivative Girl and the first n copies can lift together. Calculate G_1 and G_2, and give a closed-form expression for G_n:

$G_1 =$ $1000 + 1000(\frac{1}{3}) \approx 1333.33$

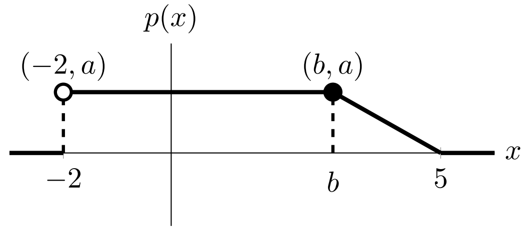
$G_2 =$ $1000+1000(\frac{1}{3})+1000(\frac{1}{9}) \approx 1444.44$

$G_n =$ $\dfrac{1000(1-(\frac{1}{3})^{n+1})}{1-1/3}$

c. [2 points] If Derivative Girl could make infinitely many copies, what is the largest amount, in kg, that Derivative Girl and her copies could lift together? Your answer should be a closed-form expression.

Answer: 1500

3. [8 points] Anya is playing a game. Each turn, Anya's score can change by x points, where x is a real number between -2 and 5. That is, her score can go up by as much as 5 points or down by as much as 2 points. The probability density function for the change in her score each turn is given by the piecewise-linear function $p(x)$ graphed below:



Here, a and b are positive constants. Do not assume the graph shown is to scale.

a. [6 points] The median amount of points Anya can score each turn is $b-1$. Find the values of a and b.

Solution: This is a pdf, so

$$\begin{aligned} 1 &= (b-(-2))a + \frac{1}{2}(5-b)a \\ &= a(\frac{1}{2}b + \frac{9}{2}) \end{aligned}$$

So $a = \frac{2}{b+9}$. The median is $b-1$, so

$$\int_{-2}^{b-1} p(x)dx = (b+1)a = \frac{1}{2}$$

Substituting for a, using the first equation, we get:

$$(b+1)\frac{2}{b+9} = \frac{1}{2}.$$

Solving, this gives us $b = \frac{5}{3}$. Substituting this into the first equation, we get $a = \frac{3}{16}$.

Answer: $a = \frac{3}{16}$ $b = \frac{5}{3}$

b. [2 points] Circle the one statement best supported by the equation

$$p(4.6) = 0.0225.$$

i) Anya will score between 4.5 and 4.7 points on about 0.45% of her turns.
ii) Anya will score 4.6 points on 2.25% of her turns.
iii) Anya will score 4.6 points on about 2.25% of her turns.
iv) Anya will score at most 4.6 points on about 2.25% of her turns.
v) Anya will score between 4.6 and 4.65 points on about 2.25% of her turns.
vi) Anya will score 0.0225 points on about 4.6% of her turns.
vii) Anya will score between 0 and 0.0225 points on about 0.1035% of her turns.

4. [8 points] The amount of time it takes a spider to build a web is t hours. The **cumulative distribution function** for t is given by:

$$J(t) = \begin{cases} 0, & \text{for } t \le \frac{1}{2} \\ \frac{16}{9}\left(-\frac{1}{3}t^3 + \frac{5}{4}t^2 - t + \frac{11}{48}\right), & \text{for } \frac{1}{2} < t < 2 \\ 1, & \text{for } 2 \le t \end{cases}$$

a. [2 points] What appears to be the shortest amount of time it could take the spider to build a web? Include units.

Answer: $\frac{1}{2}$ **hour**

b. [2 points] What is the probability that it will take the spider more than 1 hour to build a web?

Answer: $1-\frac{7}{27} \approx 0.74074$

c. [4 points] Write an expression for the mean amount of time it takes the spider to build a web. Your answer may involve one or more integrals, but should not involve the letter J.

Answer: $\displaystyle\int_{1/2}^{2} \frac{16t}{9}\left(-t^2 + \frac{5}{2}t - 1\right) dt$

5. [9 points]

a. [5 points] Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n!(3n)}{(2n)!\,3^n}(x-7)^n.$$

Show all your work.

Solution: Let $a_n = \frac{n!(3n)}{(2n)!\,3^n}(x-7)^n$. Then we will find the radius of convergence by applying the ratio test to $\sum_{n=0}^{\infty} a_n$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{\frac{(n+1)!(3n+3)}{(2n+2)!3^{n+1}}|x-7|^{n+1}}{\frac{n!(3n)}{(2n)!3^n}|x-7|^n} \\
&= \lim_{n\to\infty} \frac{(n+1)!(3n+3)(2n)!3^n|x-7|^{n+1}}{(2n+2)!3^{n+1}n!(3n)|x-7|^n} \\
&= \lim_{n\to\infty} \frac{(n+1)(3n+3)}{(2n+2)(2n+1)3(3n)}|x-7| \\
&= 0
\end{aligned}$$

for all x, since the degree of the denominator is greater than the degree of the numerator. This means that $\sum_{n=0}^{\infty} a_n$ converges for all x, and the radius is infinity.

Radius: ∞

b. [4 points] The power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{6^n\sqrt{n^2+n+7}}(x-4)^n$ has radius of convergence $R = 6$.

At which of the following x-values does the power series converge? Circle all correct answers. You do not need to justify your answer.

i. $x = -6$

ii. $x = -2$

iii. $\boxed{x = 0}$

iv. $\boxed{x = 4}$

v. $\boxed{x = 6}$

vi. $\boxed{x = 10}$

vii. $x = 12$

viii. NONE OF THESE

6. [8 points] Derivative Girl lifts a bucket of water at a constant velocity from the ground up to a platform 50 meters above the ground. The bucket and water start at a total mass of 20 kg, but while it is being lifted, a total of 3 kg of water drips out at a steady rate through a hole in the bottom of the bucket.

For this problem, you may assume that acceleration due to gravity is $g = 9.8\ \mathrm{m/s^2}$.

a. [2 points] Give an expression giving the mass of the bucket and water when the bucket is h meters above ground. Include units.

Answer: Mass of water = $20 - \frac{3}{50}h$ **kg**

b. [3 points] Suppose Δh is small. Write an expression (not involving integrals) that approximates the work required to lift the bucket from a height of h meters above the ground to a height of $h + \Delta h$ meters above the ground. Include units.

Answer: Work $\approx$ $9.8(20 - \frac{3}{50}h)\Delta h$ **joules**

c. [3 points] Write, but do not evaluate, an integral that gives the work required to lift the bucket from the ground to the platform. Include units.

Answer: $\int_0^{50} 9.8(20 - \frac{3}{50}h)\, dh$ **joules**

7. [6 points] Determine whether the following series converges or diverges. Be sure to fully justify your answer, showing all work and indicating any theorems you use.

$$\sum_{n=1}^{\infty} \frac{(n-1)\left(\cos^2(n)\right)}{7n^6+2n^4+n}$$

Answer *(Circle one):* **Diverges** $\boxed{\textbf{Converges}}$

Justification:

Solution: Notice

$$\frac{(n-1)\left(\cos^2(n)\right)}{7n^6+2n^4+n} \leq \frac{n}{7n^6+2n^4+n} < \frac{n}{7n^6} = \frac{1}{7n^5}.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{7n^5}$ converges, as it is a $p-$series, with $p = 5 > 1$. So, by the (direct) comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)\left(\cos^2(n)\right)}{7n^6+2n^4+n}$ converges.

8. [8 points] Determine whether the following series converges absolutely, converges conditionally, or diverges. Be sure to fully justify your answer, showing all work and indicating any theorems you use.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n^2+1}}$$

(Circle one): **Converges Absolutely** $\boxed{\textbf{Converges Conditionally}}$ **Diverges**

Solution: We have $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n+\sqrt{n^2+1}}\right| = \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n^2+1}}$. Notice

$$\lim_{n\to\infty} \frac{\frac{1}{n+\sqrt{n^2+1}}}{\frac{1}{n}} = \frac{1}{2},$$

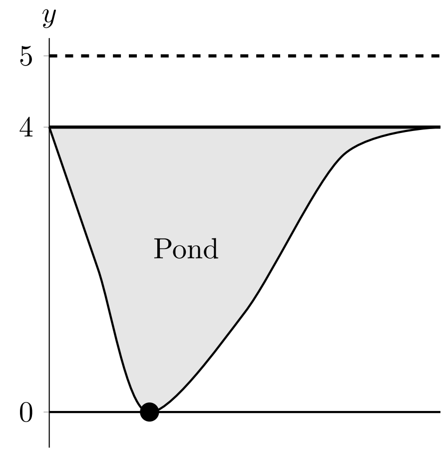
so since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since it is a $p-$series, with $p=1$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n^2+1}}$ diverges by limit comparison test. This tells us that the series cannot converge absolutely, but it may still converge conditionally.

So we need to check convergence of the series without absolute value. Since this is an alternating series, $\displaystyle\lim_{n\to\infty} \frac{1}{n+\sqrt{n^2+1}} = 0$, and $\displaystyle\frac{1}{n+\sqrt{n^2+1}}$ is decreasing, the series satisfies the hypotheses of the Alternating Series Test, and therefore we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n^2+1}}$ converges.

9. [9 points]

A small pond has murky water, and needs to be completely drained.

- A side view of the pond looks like the diagram at right.
- y measures the distance, in meters, above the bottom of the pond.
- The surface of the pond is at $y = 4$.
- The water must be pumped to a height **1 meter above the surface.**
- The cross-sections perpendicular to the y-axis are **circles.**
- The **radius** of the circular cross-section y meters above the bottom of the pond is $r(y)$ meters.



- The **density** of the murky water varies with y, and is given by $Q(y)$ kg/m^3.
- Note that the domain for both r and Q is $[0, 4]$.
- You may assume that acceleration due to gravity is $g = 9.8$ m/s^2.

Note that your answers below may include $r(y)$ and $Q(y)$.

a. [3 points] Write an expression that gives the approximate mass, in kilograms, of a slice of the murky water that is Δy m thick and at a height of y meters. Your expression should not involve any integrals.

Answer: Mass of slice ≈ $\pi r(y)^2 Q(y)\Delta y$ **kg**

b. [3 points] Write an expression in terms of y that approximates the work, in joules, done in pumping a horizontal slice of murky water of thickness Δy at a height of y meters to 1 meter above the surface of the pond. Your expression should not involve any integrals.

Answer: Work ≈ $9.8(5 - y)\pi r(y)^2 Q(y)\Delta y$ **Joules**

c. [3 points] Write an expression involving one or more integrals that gives the total work, in joules, to completely drain the pond by pumping all the water to 1 meter above the pond.

Answer: $\int_0^4 9.8(5 - y)\pi r(y)^2 Q(y)dy$ **Joules**

10. [9 points]

a. [3 points] Let $\sum_{n=1}^{\infty} a_n$ be a series, and let $S_j = a_1 + a_2 + \cdots + a_j$ be the partial sum of the first j terms of the series. If $S_j = \dfrac{4}{1+\frac{1}{j}}$, which of the following statements **must** be true? Circle all correct answers.

i. $\sum_{n=1}^{\infty} a_n$ diverges

ii. $\boxed{\sum_{n=1}^{\infty} a_n \text{ converges}}$

iii. $\boxed{\sum_{n=1}^{\infty} a_n = 4}$

iv. $\boxed{\text{the sequence } a_n \text{ converges}}$

v. $\boxed{\text{the sequence } S_j \text{ converges to } 4}$

vi. the sequence S_j diverges

vii. $\boxed{a_n = \dfrac{4}{1+\frac{1}{n}} - \dfrac{4}{1+\frac{1}{n-1}} \text{ for } n \geq 2}$

viii. NONE OF THESE

b. [3 points] Let $h(x)$ be a positive, continuous, decreasing function such that $\int_1^{\infty} h(x)\,dx = 32$, and let $b_n = h(n)$. Which of the following **must** be true? Circle all correct answers.

i. $\sum_{n=1}^{\infty} b_n$ diverges

ii. $\boxed{\sum_{n=1}^{\infty} b_n \text{ converges}}$

iii. $\sum_{n=1}^{\infty} b_n = 32$

iv. $\boxed{\sum_{n=1}^{\infty} (-1)^n b_n \text{ converges}}$

v. NONE OF THESE

c. [3 points] The force required to compress a spring by a distance of x meters from its equilibrium position is given by $F = kx$, for some constant k, measured in newtons/meter. Let $p(k)$ be the probability density function of the value of k of a batch of springs.

Which of the following represents the probability that the force for compressing a spring in this batch 0.1 m from its equilibrium position is between 0.4 and 0.6 newtons? Circle the one best answer.

i. $\boxed{\int_4^6 p(k)\,dk}$

ii. $p(6) - p(4)$

iii. $\int_{0.4}^{0.6} p(k)\,dk$

iv. $p(0.6) - p(0.4)$

v. $p(120) - p(80)$

vi. NONE OF THESE

11. [15 points] Let a_n, b_n, and c_n be sequences such that

- $a_n > 0$ for all n,
- $\sum_{n=1}^{\infty} a_n$ diverges,
- $a_n < b_n < n$ for all n,
- $3 < c_n < 4$ for all n.

Determine whether each of the following must diverge, must converge, or if there is not enough information to decide.
If you say that a series must converge or diverge:

- **name or state a test** that can be used to justify your claim.
- If you use the direct or limit comparison test, also **name the comparison series** you would use.
  You do *not* need to justify the convergence or divergence of your comparison series.

a. [3 points] $\sum_{n=1}^{\infty} b_n$

Answer *(circle one)*: $\boxed{\text{DIVERGES}}$ CONVERGES or NOT ENOUGH INFO

Solution: Comparison test, $\sum_{n=1}^{\infty} a_n$.

b. [3 points] $\sum_{n=1}^{\infty} \frac{1}{b_n}$

Answer *(circle one)*: $\boxed{\text{DIVERGES}}$ CONVERGES or NOT ENOUGH INFO

Solution: Comparison test, $\sum_{n=1}^{\infty} \frac{1}{n}$.

c. [3 points] $\sum_{n=1}^{\infty} c_n$

Answer *(circle one)*: $\boxed{\text{DIVERGES}}$ CONVERGES or NOT ENOUGH INFO

Solution: The n^{th} term test for divergence, or comparison test with $\sum_{n=1}^{\infty} 3$.

d. [3 points] $\sum_{n=1}^{\infty} \frac{1}{(c_n)^n}$

Answer *(circle one)*: DIVERGES $\boxed{\text{CONVERGES}}$ or NOT ENOUGH INFO

Solution: Comparison test, $\sum_{n=1}^{\infty} \frac{1}{3^n}$.

e. [3 points] $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{a_n}$

Answer *(circle one)*: DIVERGES CONVERGES or $\boxed{\text{NOT ENOUGH INFO}}$