Math 116 — Final Exam — December 17, 2019

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. Do not separate the pages of this exam. If pages do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
4. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	9	
2	9	
3	10	
4	10	
5	13	

Problem	Points	Score
6	11	
7	11	
8	9	
9	9	
10	9	
Total	100	

1. [9 points] A ball of dough is left on a counter to rise. Its height above the table t minutes after being left out is H centimeters. The ball's height satisfies the differential equation

$$\frac{dH}{dt} = t(20 - H).$$

a. [3 points] Find all equilibrium solutions of the differential equation, and classify them as either stable or not stable.

Stable: $H = 20$

Not stable: ______

b. [6 points] When the ball of dough is first left out, it is 10 centimeters tall. Find an explicit formula for $H(t)$. Show all work.

Solution: Separating variables, we have:

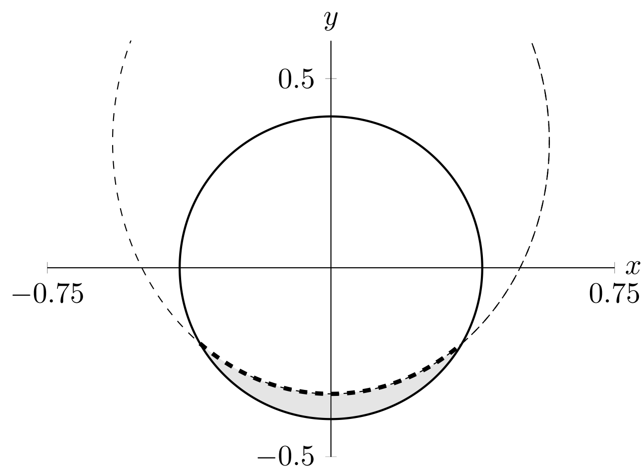
$$\frac{dH}{dt} = t(20 - H)$$
$$\int \frac{1}{20 - H} dH = \int t dt$$
$$-\ln|20 - H| = \frac{1}{2}t^2 + C_1$$
$$|20 - H| = e^{-C_1}e^{-\frac{t^2}{2}}$$
$$20 - H = Ce^{-\frac{t^2}{2}}$$
$$H = 20 - Ce^{-\frac{t^2}{2}}$$

At $t = 0$, $H = 10$, so $10 = 20 - Ce^{-\frac{0}{2}}$, and $C = 10$.

$H(t) =$ $20 - 10e^{\frac{-t^2}{2}}$

2. [9 points] Scientists are studying the bite of several different rodents. To do this, they give a wafer cookie to the animal, and take it away after the animal takes one bite.

- r is measured in inches
- The **wafer** is modeled by the region inside the polar curve
$$r = \frac{2}{5}$$
(the **solid** line in the diagram).
- The rodent's **bite** is modeled by the region inside the polar curve
$$r = \frac{1}{2 - \sin(\theta)}$$
and inside the wafer (the **dashed** line in the diagram).
- The wafer remaining after the bite is shaded in the diagram.



a. [3 points] For what values of θ between 0 and 2π does the rodent's bite meet the edge of the wafer? Justify your answer algebraically, and give your answers in **exact** form.

Solution: The bite meets the edge of the wafer when

$$\frac{2}{5} = \frac{1}{2 - \sin(\theta)},$$

which happens when $\sin(\theta) = \frac{-1}{2}$, giving us $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.
Note that $\arcsin(-1/2) < 0$, so this does not satisfy the requirements of the problem. However, we could use this to find that $\pi - \arcsin(-1/2)$ and $2\pi + \arcsin(-1/2)$ are both between 0 and 2π.

Answer: $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

b. [3 points] Write, but do not evaluate, an expression involving one or more integrals that gives the area, in square inches, of the wafer remaining after the bite.

Answer: $\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{4}{25} - \frac{1}{(2 - \sin(\theta))^2}\, d\theta$

c. [3 points] The bite mark in the wafer is represented by the **thick** dashed line in the diagram. Write, but do not evaluate, an expression involving one or more integrals that gives the length, in inches, of this bite mark.

Solution: We use the formula for arc length and

$$\frac{dr}{d\theta} = \frac{\cos(\theta)}{(2 - \sin(\theta))^2}.$$

Answer: $\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \sqrt{\frac{1}{(2 - \sin(\theta))^2} + \frac{\cos^2(\theta)}{(2 - \sin(\theta))^4}}\, d\theta$

3. [10 points] The Taylor series centered at 3 for a function $g(x)$ is given by

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^{2n}}{n\,4^n}.$$

a. [5 points] Determine the radius of convergence for this Taylor series. Show all work.

Solution: Let $a_n = \frac{(-1)^n}{n\,4^n}(x-3)^{2n}$. Then we will find the radius of convergence by applying the ratio test to $\sum_{n=1}^{\infty} a_n$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{\frac{1}{(n+1)4^{n+1}}|x-3|^{2n+2}}{\frac{1}{n4^n|x-3|^{2n}}} \\
&= \lim_{n\to\infty} \frac{n}{4(n+1)}|x-3|^2 \\
&= \frac{1}{4}|x-3|^2
\end{aligned}$$

This is less than 1 when $|x-3|^2 < 4$, or, equivalently, when $|x-3| < 2$. This means that $\sum_{n=0}^{\infty} a_n$ converges when $|x-3| < 2$, and the radius is 2.

Radius: 2

b. [2 points] Which of the following best describes the concavity of $g(x)$ at $x = 3$? Circle the <u>one</u> best answer. No justification is necessary.

CONCAVE UP $\boxed{\text{CONCAVE DOWN}}$ NEITHER   CANNOT BE DETERMINED

c. [3 points] Find $g^{(1010)}(3)$.

Solution: Using the formula for Taylor series, we know that the degree 1010 term will be $\frac{g^{(1010)}(3)}{1010!}(x-3)^{1010}$. With the formula we've been given for this problem, we see that the degree 1010 term happens when $n = 1010/2 = 505$, and will be $\frac{-(x-3)^{1010}}{505 \cdot 4^{505}}$. Therefore

$$\begin{aligned}
\frac{g^{(1010)(3)}}{1010!}(x-3)^{1010} &= \frac{-(x-3)^{1010}}{505 \cdot 4^{505}} \\
\frac{g^{(1010)}(3)}{1010!} &= -\frac{1}{505 \cdot 4^{505}} \\
g^{(1010)}(3) &= -\frac{1010!}{505 \cdot 4^{505}}
\end{aligned}$$

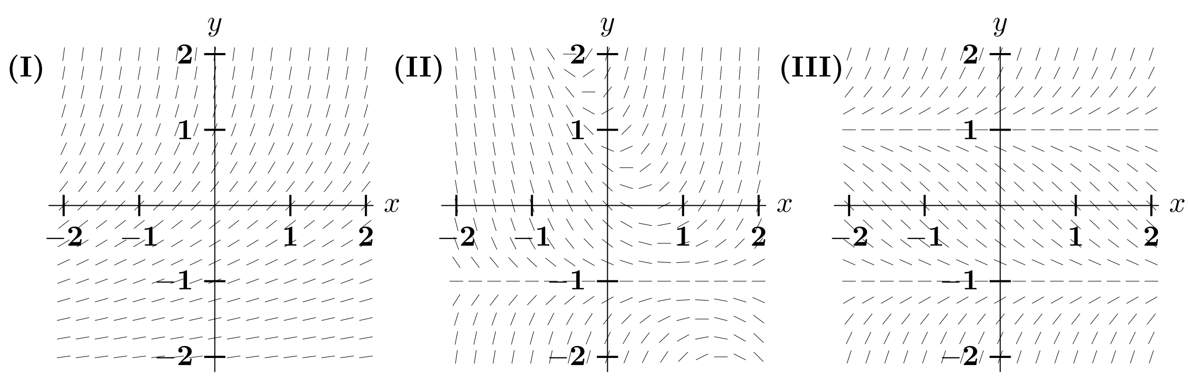
$g^{(1010)}(3) =$ $-\dfrac{1010!}{505 \cdot 4^{505}}$

4. [10 points] The following parts are unrelated.

a. [4 points] In the blockbuster movie adaptation of Derivative Girl, Darth Integrator adds toxin into a lake. The lake holds 5000 kiloliters of liquid, and when the toxin is added, it instantly mixes into the lake. A stream takes liquid from the lake at a constant rate of 100 kiloliters per day, and the toxin is added at a rate of 100 kiloliters per day. Let $L(t)$ model the amount of toxin in the lake on day t, where L is measured in kiloliters. Write a differential equation for L that models the scenario.

Answer: $\dfrac{dL}{dt} =$ $100 - 100\left(\frac{L}{5000}\right)$

b. [6 points] Consider the three slope fields shown below.



For each of the properties below, circle <u>all</u> of the slope fields for which the corresponding differential equation appears to satisfy that property. Circle NONE OF THESE if none of the differential equations satisfy the property. No justification necessary.

(i) $y = -1$ is a stable equilibrium solution.

I II $\boxed{\text{III}}$ NONE OF THESE

(ii) $y = e^x$ could be a solution to the differential equation.

I II III $\boxed{\text{NONE OF THESE}}$

(iii) The differential equation can be written in the form $\dfrac{dy}{dx} = f(y)$ for some function f.

$\boxed{\text{I}}$ II $\boxed{\text{III}}$ NONE OF THESE

5. [13 points] Two particles move in the xy-plane. At time $t > 0$, the position of particle A is given by

$$\begin{cases} x(t) = -6 - 3t \\ y(t) = 2t - k \end{cases}$$

and the position of particle B is

$$\begin{cases} x(t) = -4t \\ y(t) = t^2 - t - 8 \end{cases}$$

For the following questions, justify your answers algebraically.

a. [4 points] Find k so that the two particles must collide.

Solution: The two particles have the same x coordinate when $-6-3t = -4t$, so when $t = 6$. To ensure they collide, at $t = 6$, they must have the same y coordinate, so:

$$2(6) - k = 6^2 - 6 - 8,$$

giving us $k = -10$.

Answer: $k =$ -10

b. [3 points] At the time the particles collide, which is moving faster?

Solution: We have $x'(t) = -3$ and $y'(t) = 2$ for particle A; $x'(t) = -4$ and $y'(t) = 2t - 1$ for particle B. So, the speed at $t = 6$ for particle A is $\sqrt{9+4} = \sqrt{13}$, and for particle B is $\sqrt{16 + (2(6)-1)^2} = \sqrt{137}$.

Answer: PARTICLE A $\boxed{\text{PARTICLE } B}$

c. [3 points] Use MID(2) to approximate the length of the path traveled by particle B between $t = 0$ and $t = 4$. Write out all the terms in your sum.

Solution: Our integral is $\int_0^4 \sqrt{16 + (2t-1)^2}dt$.

Answer: $2\left(\sqrt{16 + (2(1)-1)^2}\right) + 2\left(\sqrt{16 + (2(3)-1)^2}\right)$

d. [3 points] For what positive values of t is the slope of the tangent line to the path of particle B positive?

Solution: The slope of the tangent line at t is $\frac{2t-1}{-4}$, which is positive when $2t - 1$ is negative, which is when $t < \frac{1}{2}$.

Answer: $0 < t < \frac{1}{2}$

6. [11 points]

a. [7 points] Determine whether the following series converges or diverges. Be sure to fully justify your answer, showing all work and indicating any theorems you use.

$$\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^3+5}$$

Answer *(Circle one)*: **Diverges** **Converges**

Justification:

Solution: We have $\dfrac{|\sin(2n)|}{n^3+5} \leq \dfrac{1}{n^3}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by $p-$test (as $p = 3 > 1$), $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin(2n)|}{n^3+5}$ converges by comparison test. So $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^3+5}$ converges absolutely.

b. [4 points] Let $f(x)$ be a positive, decreasing function on $[1, \infty)$ with $\lim_{x\to\infty} f(x) = 1$, and let $a_n = f(n)$ and $S_n = a_1 + \cdots + a_n$ for all $n \geq 1$.
Decide whether the following converge, diverge, or if it cannot be determined. No justification is necessary.

(i) The integral $\displaystyle\int_1^{\infty} f(x)\, dx$

$\boxed{\textbf{Diverges}}$ **Converges** CANNOT BE DETERMINED

(ii) The sequence a_n

Diverges $\boxed{\textbf{Converges}}$ CANNOT BE DETERMINED

(iii) The sequence S_n

$\boxed{\textbf{Diverges}}$ **Converges** CANNOT BE DETERMINED

(iv) The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$

$\boxed{\textbf{Diverges}}$ **Converges** CANNOT BE DETERMINED

7. [11 points] Some values of a function $m(x)$ and its derivatives are given below.

x	0	2
$m(x)$	4	1
$m'(x)$	-1	0
$m''(x)$	0	0
$m'''(x)$	3	-2
$m''''(x)$	5	8

a. [4 points] Find a formula for $P_4(x)$, the Taylor polynomial of degree 4 for $m(x)$ about $x = 2$.

Answer: $P_4(x) = $ $1 - \frac{2}{3!}(x-2)^3 + \frac{8}{4!}(x-2)^4$

b. [3 points] Use your answer to approximate the value of $\int_1^3 m(x)\,dx$. Show your work.

Solution:

$$\int_1^3 1 - \frac{2}{3!}(x-2)^3 + \frac{8}{4!}(x-2)^4 dx = x - \frac{2}{4!}(x-2)^4 + \frac{8}{5!}(x-2)^5\Big|_1^3$$
$$= (1 - \frac{2}{4!} + \frac{8}{5!}) - (-1 - \frac{2}{4!} - \frac{8}{5!})$$
$$= 2 + \frac{16}{5!} = \frac{32}{15}$$

Answer: $\int_1^3 m(x)\,dx \approx$ $\frac{32}{15}$

c. [4 points] Let $G(x)$ be the antiderivative of the function $g(x) = m(3x^2)$ with $G(0) = 5$. Find the first three nonzero terms of the Taylor series for $G(x)$ about $x = 0$.

Solution:
Using the table, we see that the first 3 nonzero terms of the Taylor series for $m(x)$ about $x = 0$ are $4 - x + \frac{1}{3!}x^3$. Then the first 3 nonzero terms for the Taylor series for $m(3x^2)$ about $x = 0$ are $4 - 3x^2 + \frac{1}{3!}(3x^2)^3$. To get the Taylor series for an antiderivative of $m(3x^2)$ about $x = 0$, we take an antiderivative of the Taylor series we found above: $C + 4x - \frac{3}{3}x^3$. Since $G(0) = 5$, we must have $C = 5$.

Answer: $5 + 4x - x^3$

8. [9 points] Derivative Girl and Gradi-Ant are excited for the end of the semester. To celebrate, they decide to make an Infinite Party Horn. In this problem, x and y are measured in meters. (In Derivative Girl's world, infinite objects are possible.)

a. [4 points] They decide to make the horn by rotating the region bounded by the positive x-axis, the positive y-axis, and the function $y = \dfrac{1}{2(x+1)^2}$ about the line $y = -1$. Write, but do not evaluate, an expression involving one integral that gives the volume, in cubic meters, of the Infinite Party Horn.

Answer: $\pi \int_0^\infty \left(\frac{1}{2(x+1)^2} + 1\right)^2 - 1\,dx$

b. [5 points] Derivative Girl will use her favorite continuous and differentiable functions f and g to make a banner for the Infinite Party Horn. She loves the functions f and g because they have the properties:

- $\dfrac{d}{dx}\left(\dfrac{1+x}{g(x)}\right) = f(x)$,
- $g(1) = 15$,
- $\lim_{x\to\infty} g(x) = \infty$,
- $\lim_{x\to\infty} g'(x) = 5$,

and the area of the banner, in square meters, is given by

$$\int_1^\infty 20f(x)\,dx.$$

Does the banner have finite area? If so, what is the banner's area? Show all work and indicate any theorems you use.

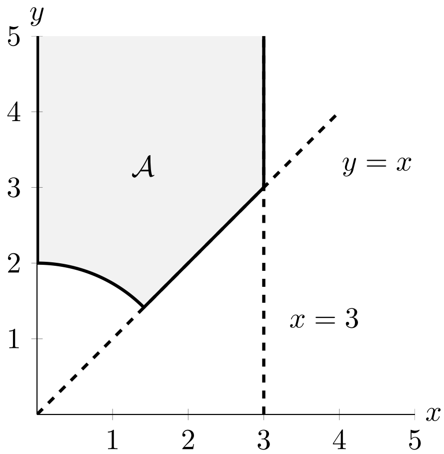
Solution: Using the information we've been given, we find

$$\begin{aligned}\int_1^\infty 20f(x)dx &= \lim_{b\to\infty}\int_1^b 20f(x)dx \\ &= \lim_{b\to\infty}\frac{20+20b}{g(b)} - \frac{40}{g(1)} \\ &= \lim_{b\to\infty}\frac{20}{g'(b)} - \frac{40}{15} \text{ by L'Hopital's Rule} \\ &= \frac{20}{5} - \frac{40}{15}\end{aligned}$$

Answer *(Circle one)*: **Infinite area** $\boxed{\textbf{Finite area:}}$ $\frac{4}{3}$ m^2

9. [9 points] The following problems are unrelated.

a. [4 points] Let $\mathcal{A}$ be the region graphed below, bounded by the y-axis, the curve $r = 2$ and lines $y = x$, $x = 3$. Give inequalities for r and θ which describe $\mathcal{A}$. Your inequalities for r may be in terms of θ. No justification is necessary.

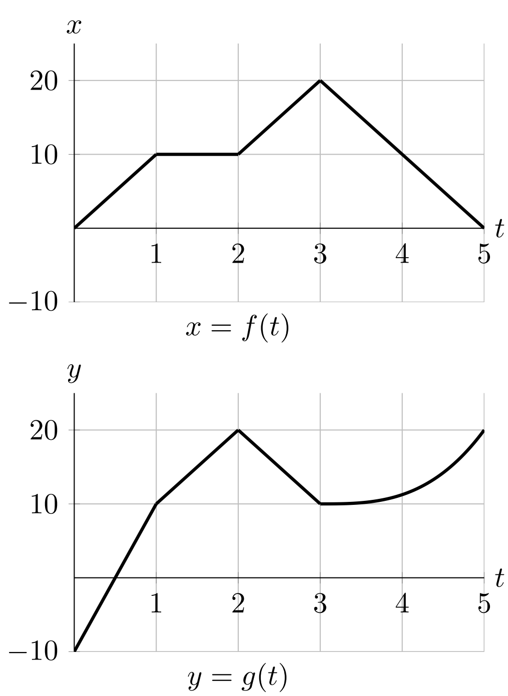


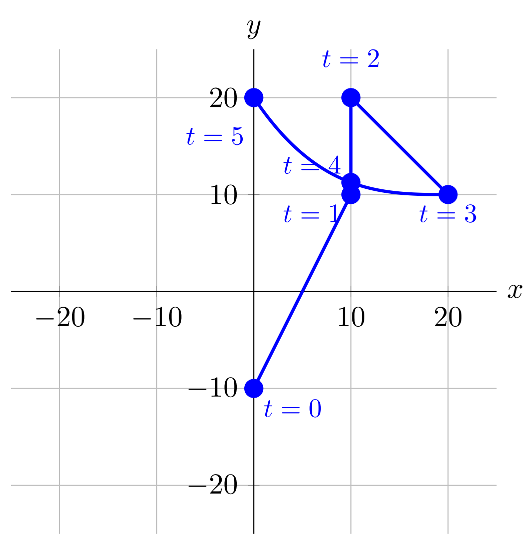
Answer:

$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$2 \le r \le \frac{3}{\cos(\theta)}$

b. [5 points] A turtle travels along the path given by the parametric equations $x = f(t)$ and $y = g(t)$ for $0 \le t \le 5$. Graphs of the functions $f(t)$ and $g(t)$ are given below.
On the axes below, sketch a graph of the path along which the turtle moves between time $t = 0$ and $t = 5$. Label the points corresponding to the position of the turtle at times $t = 0, 1, 2, 3, 4$, and 5.





10. [9 points] For each of the following, circle <u>all</u> correct answers. No justification is necessary.

a. [3 points] The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}((2x)^{n+1})}{n}$ is a Taylor series centered at $x = 0$ for some function $q(x)$. Which of the following statements are true?

i. The series diverges at $x = \frac{1}{2}$.

ii. $\boxed{\text{The series diverges at } x = \frac{-1}{2}.}$

iii. $\boxed{q(x) \text{ could be } 2x\ln(1+2x).}$

iv. $q(x)$ could be $\ln(1 + 2x)$.

v. The series converges to $6\ln(1 + 6)$ at $x = 3$.

vi. The series converges to $\ln(1 + 6)$ at $x = 3$.

vii. NONE OF THESE

b. [3 points] Let $f(x)$ be a continuous, differentiable function, with $f(1) = 1$, $f'(1) = 2$. Which of the following must be an antiderivative of $f'(f(x))f'(x)$ that passes through $(1, 3)$?

i. $\boxed{f(f(x)) + 2}$

ii. $f'(f(x)) + 1$

iii. $\dfrac{1}{2}(f(x))^2 + \dfrac{5}{2}$

iv. $3 + \displaystyle\int_0^{x-1} f'(f(t))f'(t)dt$

v. $\displaystyle\int_1^x f'(f(t))f'(t)dt$

vi. $\boxed{3 + \displaystyle\int_0^{x-1} f'(f(t+1))f'(t+1)dt}$

vii. $\boxed{3 + \displaystyle\int_1^{x} f'(f(t))f'(t)dt}$

viii. NONE OF THESE

c. [3 points] Which of the following functions are solutions to the differential equation $\dfrac{d^2y}{dx^2} + 4y = 0$?

i. $y = 4\cos(x)$

ii. $\boxed{y = \cos(2x)}$

iii. $y = \cos(2x) + 4$

iv. $\boxed{y = 4\cos(2x)}$

v. $y = e^{2x}$

vi. $y = e^{-2x}$

vii. NONE OF THESE

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Normal Distributions

The density function of a normal distribution with mean μ and standard deviation $\sigma > 0$ is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

The standard normal distribution is the normal distribution with $\mu = 0$ and $\sigma = 1$.