

Math 116 — Final Exam — December 10, 2020

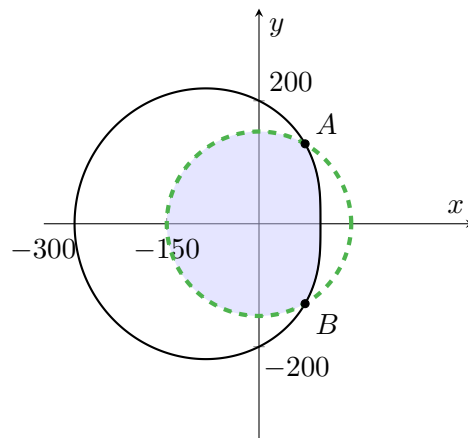
1. **Do not write your name anywhere on this exam or your submissions.**
 2. Your submissions must, however, have the correct problem number (if using blank paper) and your UMID and initials in the top corner of every page of every submission.
 3. You must submit your work for each problem in Gradescope in the appropriate location.
 4. Your submissions must be legible and rotated correctly.
 5. This exam has 9 pages including this cover.
 6. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
 7. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
 8. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
 9. You may not communicate with any other individuals, message boards, forums, etc., either in person or virtually, while you are taking the midterm.
 10. You may not use a calculator, phone, computer, tablet, laptop, or any other electronic device to assist you with any question.
You are allowed any handwritten or printed notes, provided they are printed prior to your beginning this exam.
 11. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
 12. Include units in your answer where that is appropriate.
 13. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
 14. You must use the methods learned in this course to solve all problems.
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Problem	Points	Score
1	7	
2	15	
3	19	
4	16	
5	12	
6	11	

Problem	Points	Score
7	5	
8	6	
9	9	
Total	100	

2. [15 points] With a crashing stork market, the infinite trumpet glitch, and the forestry expansion over-expanding, the video game *Vegetable Crossing* has a lot of issues. Maria designs a new island for the game, and on the island there is an area where players can grow acacia plants.

- The island is in the shape of the polar curve $r = 200 - 100 \cos(\theta)$ where $0 \leq \theta < 2\pi$. The outline of the island is the **solid black curve** plotted below.
- **The acacia-growing zone is shaded blue**, and it is formed by the section of the island inside a circle of radius 150 meters centered at the origin. The circle is the dashed green curve plotted below.
- All distances on the graph are in meters.



- a. [5 points] The points A and B , labeled above, are the intersection points of the polar curve $r = 200 - 100 \cos(\theta)$ with the dashed green circle. Find points A and B **in polar coordinates**.
- b. [5 points] Find an expression involving one or more integrals for the length, in meters, of the perimeter of the acacia-growing zone. Do not evaluate your integral(s).
- c. [5 points] Players are able to pave any part of the island **outside** of the acacia-growing zone, at a cost of 7 dubloons per square meter. Find an expression involving one or more integrals for the cost, in dubloons, of paving the entire area which lies outside of the acacia-growing zone. Do not evaluate your integral(s).

3. [19 points] Consider the function $B(x)$ described on its domain by its Taylor series around $x = 0$,

$$B(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!(2n)!} x^{2n}.$$

- a. [5 points] Find the first four non-zero terms of the Taylor series for $B(x)$ about $x = 0$. You do not need to evaluate any factorials in your answer.
- b. [6 points] Find the radius of convergence of the Taylor series. Show all of your work and use proper notation.
- c. [3 points] Is $B(x)$ an odd function, an even function, or neither? Explain your reasoning.
- d. [5 points] Find the value of $B^{(2020)}(0)$. You do not need to evaluate any factorials in your answer.

4. [16 points] Buluu and Nyekundu the giraffes work on another art project. They paint on a large, square canvas, with corners at $(0, 0)$, $(0, 100)$, $(100, 0)$, and $(100, 100)$, with distances in yards. As Buluu moves, he leaves a trail of blue paint. As Nyekundu moves, she leaves a trail of red paint.

Buluu's position t minutes after the giraffes start painting is described by the parametric equations:

$$\begin{cases} x(t) = t^2 \\ y(t) = 40 + 2(t - 5)^2 \end{cases}$$

and Nyekundu's position t minutes after the giraffes start painting is described by the parametric equations:

$$\begin{cases} x(t) = 25 + 20 \sin(\pi t) \\ y(t) = 40 - (t - 5)^2 \end{cases}$$

- a. [5 points] If the giraffes ever collide, they produce a purple splotch on the canvas at the point of collision. Will the giraffes produce any purple splotches? If so, at **what time(s)** does this occur, and **where on the canvas** are the purple splotches located?

- b. [6 points] Find an expression for Buluu's speed 3 minutes after the giraffes start painting. Make sure to include units.

- c. [5 points] At what time does Nyekundu first leave the canvas? Make sure to fully justify your answer.

5. [12 points] For each of the following, write down the CAPITAL LETTER for the **ONE** best answer for your submission. You do not need to show work, but it is strongly suggested that you try to solve the problem from start to finish before selecting your answer.

a. [4 points] A container in the shape of a cube with side length ℓ meters is resting with one face on the ground. It is filled $1/3$ of the way with a liquid of constant density 2000 kg/m^3 . Recall the gravitational constant is $g = 9.8 \text{ m/s}^2$. How much work in joules is required to pump all the liquid to the top of the container?

(A) $\int_0^{\frac{1}{3}\ell} 2000g(\ell - h)h^2 \, d\ell$

(B) $\int_0^{\frac{1}{3}\ell} 2000g(\ell - h)\ell^2 \, dh$

(C) $\int_{\frac{1}{3}\ell}^{\ell} 2000g(\ell - h)h^2 \, dh$

(D) $\int_{\frac{1}{3}\ell}^{\ell} 2000g(\ell - h)\ell^2 \, d\ell$

(E) None of the above.

b. [4 points] Suppose $p(x)$ is a probability density function for x , the total weight of avocados, **in millions of pounds**, eaten in the United States on a given day. The equation $p(1) = 0.5$ can be best interpreted as: "On any given day,...

(A)...there is a 50% chance that 1 million pounds of avocados will be eaten in the United States."

(B) ...there is roughly 100% chance that between 0 and 2 million pounds of avocados will be eaten in the United States."

(C) ...there is approximately a 10% chance that between 900,000 and 1,100,000 pounds of avocados will be eaten in the United States."

(D) ...the median weight of avocados eaten in the United States is 1 million pounds."

(E) None of the above.

c. [4 points] Consider the series

$$\sum_{n=1}^{\infty} a_n = 0 - \frac{1}{2} + 0 - \frac{1}{4} + 0 - \frac{1}{6} + 0 - \frac{1}{8} + \dots$$

which is the infinite sum of the terms in the sequence defined by

$$a_n = \begin{cases} 0 & \text{for } n \text{ odd.} \\ -\frac{1}{n} & \text{for } n \text{ even.} \end{cases}$$

The series $\sum_{n=1}^{\infty} a_n \dots$

(A) ...converges by the alternating series test.

(B) ...converges because it converges absolutely.

(C) ...converges by the ratio test.

(D) ...converges because $\lim_{n \rightarrow \infty} a_n = 0$.

(E) None of the above.

6. [11 points]

- a. [6 points] Find the Taylor series about $x = 0$ for the function $f(x) = 3 + \cos(2x^2)$. Write your answer using sigma notation and also write out the first **three** non-zero terms. You do not need to simplify any factorials or exponentials that appear in your answer.

- b. [5 points] The function $f(x)$ from part a) has an antiderivative $F(x)$ which satisfies $F(0) = 9$. Find the first four nonzero terms of the Taylor series about $x = 0$ for $F(x)$. You do not need to simplify any factorials or exponentials that appear in your answer.

7. [5 points] Find an expression for the exact value of

$$12 + \frac{4}{5} - \frac{4^2}{2(5)^2} + \frac{4^3}{3(5)^3} + \cdots + \frac{(-1)^{n+1}4^n}{n5^n} + \cdots$$

which does not involve an infinite sum (i.e. no sigma notation or "...").

8. [6 points] Suppose

$$F(x) = \int_3^{2x} (e^{5t^2} - 2) dt.$$

Find all x -values where the graph of $y = F(x)$ has a horizontal tangent line, showing all of your work. You do not need to simplify your answer(s).

9. [9 points] For each part of this problem, write the CAPITAL LETTER corresponding to **ALL** answers that apply on your submission. You do not need to show your work.

- a. [5 points] Suppose $f(x)$ is a continuous function defined for $x \geq 1$ satisfying:

- $f(x) > 0$ for all $x \geq 1$.
- $f(x)$ is decreasing on its domain.
- $f(x) \leq \frac{1}{\sqrt{x}}$

Which of the following MUST be true about $f(x)$?

- (A) $\int_1^{\infty} f(x) dx$ converges.
- (B) $\int_1^{\infty} (f(x))^2 dx$ converges.
- (C) $\int_1^{\infty} \frac{f(x)}{x} dx$ converges.
- (D) $\int_1^{\infty} f(x) dx$ diverges.
- (E) None of the above.

- b. [4 points] Which of the following pairs of polar coordinates are the same point in the xy -plane as the point $(x, y) = (-1, 1)$?

- (A) $(r, \theta) = \left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$
- (B) $(r, \theta) = \left(1, \frac{3\pi}{4}\right)$
- (C) $(r, \theta) = \left(-\sqrt{2}, -\frac{\pi}{4}\right)$
- (D) $(r, \theta) = \left(-\frac{\sqrt{2}}{2}, \frac{5\pi}{4}\right)$
- (E) None of the above.

“Known” Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \quad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \quad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \quad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \quad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots \quad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \quad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin \theta$	$\cos \theta$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$