Math 116 — Midterm — October 5, 2020

1. **Do not write your name anywhere on this exam or your submissions.**
2. Your submissions must, however, have the correct problem number (if using blank paper) and your UMID and initials in the top corner of <u>every</u> page of <u>every</u> submission.
3. You <u>must</u> submit your work for each problem in Gradescope in the appropriate location.
4. Your submissions <u>must</u> be legible and rotated correctly.
5. This exam has 13 pages including this cover.
6. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
7. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
8. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
9. You may <u>not</u> communicate with any other individuals, message boards, forums, etc., either in person or virtually, while you are taking the midterm.
10. You may not use a calculator, phone, computer, tablet, laptop, or any other electronic device to assist you with any question.

    You are allowed any handwritten or printed notes, provided they are printed prior to your beginning this exam.
11. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
12. Include units in your answer where that is appropriate.
13. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	14	
2	11	
3	6	
4	5	
5	15	
6	10	

Problem	Points	Score
7	15	
8	8	
9	16	
Total	100	

1. [14 points] Let $f(x)$ be a differentiable function whose derivative $f'(x)$ is also differentiable. Some values of $f(x)$ and $f'(x)$ are given in the table below:

x	0	1	2	3	4	5
$f(x)$	5	0	4	$\pi/2$	2	7
$f'(x)$	0	π	8	13	1	-6

Additionally, assume $f(x)$ is positive for $x > 2$.
Compute the exact value of the following integrals. If it is not possible to do so based on the information provided, then write 'NOT POSSIBLE' and clearly indicate why it is not possible. Show all of your work.

a. [4 points] $\displaystyle\int_1^2 \frac{f'(3x-1)}{f(3x-1)}\,dx$

Solution: Substitute $w = f(3x-1)$ to obtain

$$\int_{f(2)}^{f(5)} \frac{1}{3w}\,dw = \frac{1}{3}\ln|w|\Big|_4^7 = \frac{1}{3}\left(\ln(7)-\ln(4)\right) = \frac{1}{3}\ln\left(\frac{7}{4}\right)$$

b. [5 points] $\displaystyle\int_0^4 (x+1)f''(x)\,dx$

Solution: Integrating by parts,

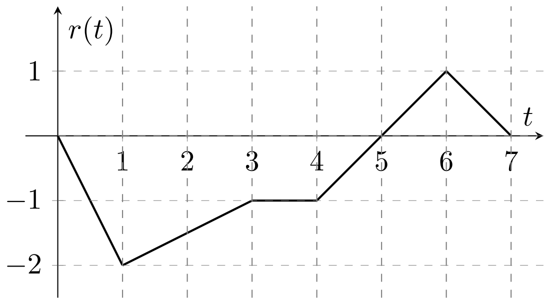
$$\begin{aligned}\int_0^4 (x+1)f''(x)\,dx =& \ (x+1)f'(x)\Big|_0^4 - \int_0^4 f'(x)\,dx\\ =& \ \left((x+1)f'(x)-f(x)\right)\Big|_0^4\\ =& \ 5f'(4)-f'(0)-f(4)+f(0)\\ =& \ 5-2+5=8\end{aligned}$$

c. [5 points] $\displaystyle\int_1^3 \left(\sin\left(f(x)\right)\right)^3\cos\left(f(x)\right)f'(x)\,dx$

Solution: Either use the substitution $w=\sin(f(x))$ or $u=f(x)$ and then $w=\sin(u)$ to obtain

$$\begin{aligned}\int_{f(1)}^{f(3)} (\sin(u))^3\cos(u)\,du = \int_{\sin(f(1))}^{\sin(f(3))} w^3\,dw \ &= \frac{1}{4}w^4\Big|_{\sin(f(1))}^{\sin(f(3))}\\ &= \tfrac{1}{4}\left(\sin^4(\pi/2)-\sin^4(0)\right)\\ &= \tfrac{1}{4}\end{aligned}$$

2. [11 points] In the game of *Vegetable Crossing*, Tina is carefully monitoring the stork market, which determines the price of a stork in dubloons, the game's currency. If t is the number of days since Tina started playing, then $r(t)$, measured in dubloons per day, gives the **rate of change** of the price of a stork in the game. A graph of $r(t)$ is shown below. Note that $r(t)$ is piecewise linear.



a. [2 points] For what value of t in $[0, 7]$ is the price of a stork growing fastest?

Solution: This will occur when $r(t)$ is at a maximum, so $t = 6$.

b. [2 points] Tina wants to buy storks when the price is as low as possible. For what value of t in $[0, 7]$ should she buy storks?

Solution: This will occur when the signed area between $r(t)$ and the t-axis is at a minimum, so $t = 5$.

c. [3 points] What is the average value of $r(t)$ on the interval $[3, 5]$? Be sure to write down any integrals you use to obtain your answer.

Solution: The average value of $r(t)$ on $[3, 5]$ is

$$\frac{1}{5-3}\int_3^5 r(t)dt.$$

Counting boxes using the grid, the integral has value -1.5, so the average value is -0.75.

d. [4 points] Let $R(t)$ be the price of a stork in dubloons at time t, and assume that $R(t)$ is continuous. The price of a stork at time $t = 3$ is 14 dubloons. Given that information, fill out the following table of values:

t	0	2	4	6
$R(t)$	18	$14 + \frac{5}{4}$	13	13

Solution: We get the values in the table by adding or subtracting the appropriate areas from 14, as we move toward or away from the t-value $t = 3$. For example, between $t = 2$ and $t = 3$, $R(t)$ decreases by $5/4$, so $R(2) = 14 + \frac{5}{4}$.

3. [6 points] Let $G(x)$ be defined by

$$G(x) = \int_{2x+3}^{5x-7} e^{t^2-1}\, dt.$$

a. [2 points] Find a value of x such that $G(x) = 0$.

Solution: This happens when both bounds are equal, so $2x + 3 = 5x - 7$. Solving this gives $x = \frac{10}{3}$.

b. [4 points] Find $G'(3)$.

Solution: We can rewrite

$$G(x) = \int_0^{5x-7} e^{t^2-1}\, dt - \int_0^{2x+7} e^{t^2-1}\, dt$$

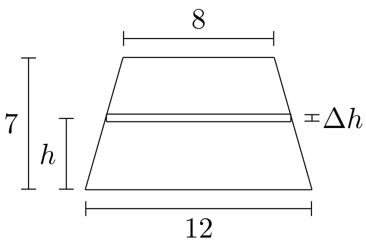
So

$$G'(x) = 5e^{(5x-7)^2-1} - 2e^{(2x+3)^2-1}$$

and so

$$G'(3) = 5e^{63} - 2e^{80}.$$

4. [5 points] A trapezoid has bases of length 12 and 8, and has height 7, as shown in the diagram below.



a. [3 points] Write an expression which approximates the area of a rectangular slice of this trapezoid with small thickness Δh at a height h from the larger base. (See the above diagram.) Your expression should not involve any integrals.

Solution: Let's call the length of slice ℓ. We can find ℓ by modeling it as a linear function of h since the sides of this object are straight.

When $h = 0$, $\ell = 12$, and when $h = 7$, $\ell = 8$. So the slope is $-4/7$, which means

$$\ell = -\frac{4}{7}h + 12,$$

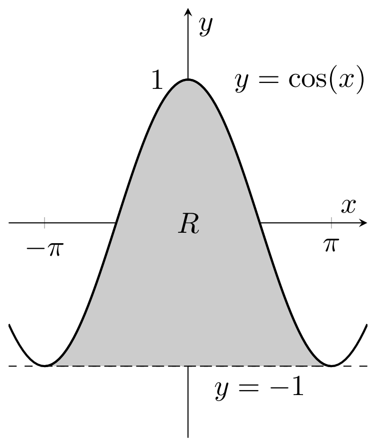
and the area of the slice is $(-\frac{4}{7}h + 12)\Delta h$.

b. [2 points] Using your expression from (a) to write an integral which, when evaluated, gives the total area of the trapezoid. Do not evaluate the integral.

Solution: The area is the integral of the area of the slice from (a) as $\Delta h \to 0$, from $h = 0$ to $h = 7$:

$$\int_0^7 \left(-\frac{4}{7}h + 12\right) dh.$$

5. [15 points] Consider the region R in the xy−plane bounded between $y = \cos(x)$ and $y = -1$ for x values between $-\pi$ and π. A sketch of the region is shown below.



a. [5 points] Find an expression involving one or more integrals for the volume of the solid formed by rotating the region R around the line $x = 5$. Do not evaluate your integral(s).

Solution: Taking vertical slices, we see that we obtain the shell method. The volume of a slice of thickness Δx at horizontal coordinate x is approximately

$$2\pi(5 - x)(\cos(x) - (-1))\Delta x$$

and so the total volume of the solid is

$$2\pi \int_{-\pi}^{\pi} (5 - x)(\cos(x) + 1)\ dx.$$

b. [5 points] Find an expression involving one or more integrals for the volume of the solid formed by rotating the region R around the line $y = -3$. Do not evaluate your integral(s).

Solution: Again using vertical slices, we obtain the washer method. The volume of a slice of thickness Δx at horizontal coordinate x is approximately

$$\pi\left((\cos(x) + 3)^2 - 2^2\right)\Delta x$$

and so the total volume of the solid is

$$\pi \int_{-\pi}^{\pi} \left((\cos(x) + 3)^2 - 4\right)\ dx.$$

c. [5 points] Find an expression involving one or more integrals for the volume of the solid with a base in the shape of the region R, and semicircular cross sections perpendicular to the x-axis. Do not evaluate your integral(s).

Solution: Again, take vertical slices. The area of a semi-circle is $\frac{1}{2}\pi r^2$ and the radius should be half the height of a slice. Putting this together, the volume of a slice of thickness Δx at horizontal coordinate x is approximately

$$\frac{1}{2}\pi\left(\frac{1}{2}(\cos(x)+1)\right)^2\Delta x$$

and so the total volume of the solid is

$$\begin{aligned} & \frac{1}{2}\pi\int_{-\pi}^{\pi}\left(\frac{1}{2}(\cos(x)+1)\right)^2\,dx \\ = \quad & \frac{\pi}{8}\int_{-\pi}^{\pi}(\cos(x)+1)^2\,dx \end{aligned}$$

6. [10 points] A thin circular plate of radius 3 m is being used to launch an electric rocket into space. The charge density, in Coulombs per m^2, on the surface of the plate a distance r meters from its center is given by a function $\delta(r) = 1 - kr$ for some constant k. Note that a Coulomb is a unit of electric charge.

a. [4 points] Write an expression involving integrals for the total charge, in Coulombs, on the surface of the circular plate. Do not evaluate the integral(s).

Solution: The charge on a thin circular ring of thickness Δr at a distance of r meters from the center is approximately

$$2\pi r(1-kr)\Delta r$$

and so the total charge on the plate is given by

$$2\pi \int_0^3 r(1-kr)\,dr.$$

b. [6 points] Find the value of k if the total charge on the surface of the plate is 3π Coulombs. Be sure to show all your work including algebra and any evaluation of integrals.

Solution: We have

$$\begin{aligned} 2\pi \int_0^3 r(1-kr)\,dr &= 3\pi \\ 2\pi \left(\tfrac{1}{2}r^2 - \tfrac{1}{3}kr^3\right)\Big|_0^3 &= 3\pi \\ 2\pi \left(\tfrac{9}{2} - 9k\right) &= 3\pi \\ 9 - 18k &= 3 \\ k &= \tfrac{1}{3}. \end{aligned}$$

7. [15 points] Consider the function

$$f(x) = \frac{1}{(x-2)^2(x-3)}$$

a. [4 points] Approximate the integral $\int_{-5}^{1} f(x)\,dx$ using MID(3). Do not decompose $f(x)$ before doing the approximation. Write out each term in any sums you make.

Solution:

$$\begin{aligned} \text{MID}(3) &= 2\left(f(-4) + f(-2) + f(0)\right) \\ &= 2\left(-\tfrac{1}{7(6^2)} - \tfrac{1}{5(4^2)} - \tfrac{1}{3(2^2)}\right) \end{aligned}$$

b. [4 points] Approximate the integral $\int_{-5}^{1} f(x)\,dx$ using TRAP(3). Do not decompose $f(x)$ before doing the approximation. Write out each term in any sums you make.

Solution:

$$\begin{aligned} \text{TRAP}(3) = &\ \frac{1}{2}(2f(-5) + 2f(-3)) + \frac{1}{2}(2f(-3) + 2f(-1)) + \frac{1}{2}(2f(-1) + 2f(1)) \\ = &\ -\frac{1}{8(7^2)} - \frac{2}{6(5^2)} - \frac{2}{4(3^2)} - \frac{1}{2(1^2)} \end{aligned}$$

Alternatively, you may take the average of LEFT(3) and RIGHT(3) to obtain the same result.

c. [7 points] Split the function $f(x)$ into partial fractions with two or more terms. Do not integrate these terms.

Solution: Let

$$\tfrac{1}{(x-2)^2(x-3)} = \tfrac{A}{(x-2)} + \tfrac{B}{(x-2)^2} + \tfrac{C}{(x-3)}.$$

Then

$$1 = A(x-2)(x-3) + B(x-3) + C(x-2)^2.$$

Comparing coefficients of x^2,

$$A + C = 0, \text{ so } C = -A.$$

Comparing coefficients of x,

$$-5A + B - 4C = 0, \text{ so } B = A.$$

Comparing coefficients of 1,

$$6A - 3B + 4C = 1, \text{ so } A = -1.$$

Substituting back, we see that $B = -1$ and $C = 1$, so

$$\frac{1}{(x-2)^2(x-3)} = \frac{-1}{(x-2)} + \frac{-1}{(x-2)^2} + \frac{1}{(x-3)}.$$

8. [8 points]

a. [4 points] Kesha is tasked with sourcing mahogany timber for her logging company. She finds that the cost of producing x tons of timber is given by $M(x)$, measured in thousands of dollars. She knows that $M'(x) = \ln(1 + x^2)$ and that it costs 11 thousand dollars to produce 3 tons of timber. Find an expression involving an integral for $M(x)$.

Solution:

$$M(x) = 11 + \int_3^x \ln(1 + t^2)\ dt$$

b. [4 points] Kesha finds that her department's profit for the month, in thousands of dollars, is given by

$$\int_{-5}^{5} \sin(x^5) + 2\ dx.$$

By evaluating this integral, give a numerical figure for Kesha's department's profit. (Hint: What special property of the function $\sin(x^5)$ could be useful here?) Make sure to fully justify your answer.

Solution:

$$\begin{aligned} \int_{-5}^{5} \sin(x^5) + 2\ dx &= \int_{-5}^{5} \sin(x^5)\ dx + \int_{-5}^{5} 2\ dx \\ &= \int_{-5}^{5} \sin(x^5)\ dx + 20 \end{aligned}$$

But $\sin(x^5)$ is an odd function since $\sin((-x)^5) = \sin(-(x^5)) = -\sin(x^5)$ (both x^5 and $\sin(x)$ are odd). This means that $\displaystyle\int_{-5}^{5} \sin(x^5)\ dx = 0$, and so the profit is 20 thousand dollars.

9. [16 points] Archaeologists are pulling artifacts from an excavation site. As a reminder, the gravitational constant g is 9.8 m/s^2.

a. [8 points] The first artifact weighs 20 kg and lies at the bottom of the excavation site. The archaeologists are 50m above the bottom and want to use a rope to pull the artifact up to a ledge 20m from the bottom of the excavation site. If the rope has mass 2 kg per meter, how much work is done pulling the first artifact from the bottom of the excavation site to the ledge? Be sure to include units and show all your work, including calculation of any integrals.

Solution: After raising the artifact x meters, the weight of the artifact and rope combined is

$$(20 + 2(50 - x))g = (120 - 2x)g \text{ newtons}$$

and so the work done in raising the artifact a short distance Δx is

$$(120 - 2x)g\Delta x \text{ joules.}$$

Therefore the total work done in raising the artifact is

$$\begin{aligned}\int_0^{20} (120 - 2x)g\ dx &= \left(120x - x^2\right) g\Big|_0^{20} \\ &= \left(120(20) - 20^2\right) g \\ &= (2400 - 400)g = 2000g \text{ joules}\end{aligned}$$

The second artifact weighs 100 lbs when the archaeologists begin lifting it 165 feet up out of the excavation site. This artifact is crumbling, and it loses weight at a constant rate of 0.2 lbs per foot it is lifted. The cable used to lift this second artifact weighs 0.5 lbs per foot.

b. [3 points] Write a formula for $w(y)$, the weight of the second artifact in lbs, when it has been lifted y feet out of the excavation site.

Solution: $w(y) = 100 - 0.2y$

c. [5 points] Write an expression involving one or more integrals for the work done pulling the second artifact out of the excavation site. Do not evaluate your integral(s).

Solution:
The weight of the artifact and the rope after it has been raised y feet is

$$w(y) + 0.5(165 - y) = 182.5 - 0.7y$$

and so the work done in raising it another Δy feet is approximately

$$(182.5 - 0.7y)\Delta y.$$

So the total work done in raising the artifact is

$$\int_0^{165} (182.5 - 0.7y)\, dy \text{ foot-pounds}$$