Math 116 — Second Midterm — November 9, 2020

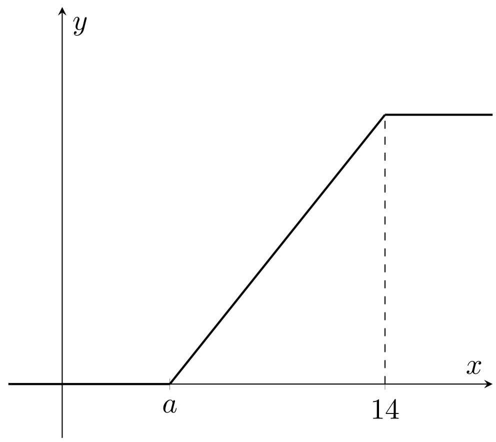
1. **Do not write your name anywhere on this exam or your submissions.**
2. Your submissions must, however, have the correct problem number (if using blank paper) and your UMID and initials in the top corner of every page of every submission.
3. You must submit your work for each problem in Gradescope in the appropriate location.
4. Your submissions must be legible and rotated correctly.
5. This exam has 11 pages including this cover.
6. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
7. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
8. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
9. You may not communicate with any other individuals, message boards, forums, etc., either in person or virtually, while you are taking the midterm.
10. You may not use a calculator, phone, computer, tablet, laptop, or any other electronic device to assist you with any question.

    You are allowed any handwritten or printed notes, provided they are printed prior to your beginning this exam.
11. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
12. Include units in your answer where that is appropriate.
13. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	8	
2	15	
3	10	
4	13	
5	14	
6	10	

Problem	Points	Score
7	16	
8	14	
Total	100	

1. [8 points] The graph of a piecewise-linear cumulative distribution function $P(x)$ is given below. The function $P(x)$ is defined for all real numbers x and is constant for $x < a$ and for $x > 14$.



a. [3 points] The median value for x is 10. Use this fact, and the fact that $P(x)$ is a cumulative distribution function to find the value of a. No justification is needed but you may earn partial credit if you show your work.

Solution: Since the median for x is 10, we must have $P(10) = 0.5$.
Since $P(x)$ is a cdf, we must have $P(14) = 1$.
The slope of $P(x)$ between a and 14 is then $\frac{0.5}{4} = \frac{1}{8}$, and by using the equation of a straight line, we see that, between a and 14, $P(x) = \frac{1}{8}x - \frac{6}{8}$. Since $P(a) = 0$, we must have $a = 6$.

b. [5 points] Find a formula for a probability density function $p(x)$ which corresponds to $P(x)$. Make sure to define your formula for all values of x, using a piecewise-defined formula if necessary. You may give your answer in terms of a.

Solution: We must have $P'(x) = p(x)$ wherever it is defined. Using our formula from part a), we see that

$$p(x) = \begin{cases} 0 & \text{for } x < 6 \\ \frac{1}{8} & \text{for } 6 < x < 14 \\ 0 & \text{for } x > 14 \end{cases}$$

2. [15 points]

a. [9 points] For each of the following sequences, defined for $n \geq 1$, state clearly whether the sequence is:

- increasing, decreasing, or neither.
- bounded or unbounded.
- convergent or divergent.

No justification is needed.

(i) $a_n = 2 - \cos(\pi n)$

Solution: neither, bounded, divergent

(ii) $b_n = \displaystyle\int_1^{n^2} \frac{1}{x}\, dx$

Solution: increasing, unbounded, divergent

(iii) $c_n = 13 - \displaystyle\sum_{k=0}^{n} \frac{1}{(1.1)^k}$

Solution: decreasing, bounded, convergent

b. [6 points] Let $\displaystyle\sum_{n=1}^{\infty} d_n$ be a geometric series, with $d_2 = 16$ and $d_5 = 2$. Determine, and clearly state, whether the series converges or diverges. If the series converges, find its sum.

Solution: Since the series is geometric, we know that $16x^3 = 2$, where x is the ratio in the series. Solving this gives $x = \frac{1}{2}$. This means the first term in the series is $a = 32$ (since $\frac{1}{2}a = 16$). Since the common ratio is $\frac{1}{2}$, the geometric series converges. Using this a and x, the sum is $32/(1 - 1/2) = 64$.

3. [10 points]

a. [6 points] Determine the **radius** of convergence for the following power series. Show all of your work. You do not need to find the interval of convergence.

$$\sum_{n=1}^{\infty}(-1)^n \frac{4^{n+1}}{n^{1/3}}(x-1)^n$$

Solution: We will use the ratio test. For $a_n = (-1)^n \frac{4^{n+1}}{n^{1/3}}(x-1)^n$, we have:

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\frac{4^{n+2}}{4^{n+1}}\frac{n^{1/3}}{(n+1)^{1/3}}\frac{(x-1)^{n+1}}{(x-1)^n}\right| \\ &= \lim_{n\to\infty} 4\frac{n^{1/3}}{(n+1)^{1/3}}|x-1| \\ &= 4|x-1|. \end{aligned}$$

By the ratio test, the power series converges when $4|x-1| < 1$, i.e. $|x-1| < \frac{1}{4}$, and so the radius of convergence is $\frac{1}{4}$.

b. [4 points] Suppose the power series $\sum_{n=0}^{\infty} C_n(x-a)^n$ has radius of convergence 2, and that the series converges for $x = 4$ and diverges for $x = 6$. Which of the following could be the value of a? List **all** correct answers.

0 1 2 3 4 5 6

Solution: The series is centered at $x = a$ and has radius of convergence 2. Since the series converges at $x = 4$, we must have $2 \le a \le 6$. Since the series diverges at $x = 6$, we cannot have $4 < a < 8$. From the list, the only possible values of a are 2, 3, and 4.

4. [13 points] Rafael finds details of another of TimberCorp's logging operations, this time in a forest of redwoods which initially has 50,000 trees. TimberCorp plans to, at the start of each year, cut down 10% of the trees in the forest, and then over the course of the year replant k trees.

a. [5 points] Let R_n be the number of trees in the forest **at the end** of the nth year of the logging operation. Find expressions for R_1 and R_2. Your answers may involve k. **You do not need to simplify your answers**.

Solution:

$$R_1 = (0.9)50000 + k$$

$$R_2 = (0.9)^2(50000) + 0.9k + k$$

b. [5 points] Find a **closed form** expression for R_n. Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive. **You do not need to simplify your closed form answer.**

Solution:

$$\begin{aligned} R_n &= (0.9)^n(50000) + k + 0.9k + (0.9)^2k + \ldots + (0.9)^nk \\ &= (0.9)^n(50000) + k\left(\tfrac{1-(0.9)^n}{1-0.9}\right) \\ &= (0.9)^n(50000) + 10k\left(1-(0.9)^n\right) \end{aligned}$$

where we used the formula for the sum of a (finite) geometric series

c. [3 points] Rafael wants the number of trees in the forest at the end of a year to tend towards 70,000 in the long run (i.e. after many many years). What value should he choose for k to ensure this happens?

Solution: As $n \to \infty$, $R_n \to 0 + 10k(1-0) = 10k$, and so R_n will tend toward 70,000 if $10k = 70000$, i.e. $k = 7000$.

5. [14 points]

a. [7 points] Determine whether the following improper integral converges or diverges. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

$$\int_0^1 \frac{3}{x + x^{1/2}}\, dx$$

Solution: On the interval $0 \le x \le 1$, we have $\frac{3}{x + x^{1/2}} \le \frac{3}{x^{1/2}}$, and $\int_0^1 \frac{3}{x^{1/2}}\, dx$ converges by the p-test with $p = \frac{1}{2}$. Therefore, by the (Direct) Comparison Test, $\int_0^1 \frac{3}{x + x^{1/2}}\, dx$ converges.

b. [7 points] **Compute** the value of the following improper integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

$$\int_1^\infty \frac{2+2e^x}{(x+e^x)^{3/2}}\,dx$$

Solution:

$$\int_1^\infty \frac{2+2e^x}{(x+e^x)^{3/2}}\,dx = \lim_{b\to\infty}\int_1^b \frac{2+2e^x}{(x+e^x)^{3/2}}\,dx.$$

Now, using the substitution $w = x+e^x$, we see

$$\int_1^b \frac{2+2e^x}{(x+e^x)^{3/2}}\,dx = \int_{1+e}^{b+e^b} \frac{2}{w^{3/2}}\,dw = -\frac{4}{w^{1/2}}\bigg|_{1+e}^{b+e^b} = \frac{4}{(1+e)^{1/2}} - \frac{4}{(b+e^b)^{1/2}}.$$

Therefore

$$\int_1^\infty \frac{2+2e^x}{(x+e^x)^{3/2}}\,dx = \lim_{b\to\infty}\left(\frac{4}{(1+e)^{1/2}} - \frac{4}{(b+e^b)^{1/2}}\right) = \frac{4}{(1+e)^{1/2}}.$$

6. [10 points] The power series

$$\sum_{n=1}^{\infty} \frac{6}{7^n\sqrt{n^2+2n}}(x-3)^n$$

has radius of convergence 7. (You can assume this to be true, and you do NOT need to verify this).

Find the **interval** of convergence for the power series. Show all your work including full justification of convergence and divergence of any relevant series.

Solution: From the form of the power series, the interval must be centered at $x = 3$. Since the radius of convergence is 7, that means the endpoints are at $x = -4$ and $x = 10$.

At $x = -4$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{6}{7^n\sqrt{n^2+2n}}(-7)^n = \sum_{n=1}^{\infty} \frac{6(-1)^n}{\sqrt{n^2+2n}}.$$

This is an alternating series, and for $a_n = \frac{6}{\sqrt{n^2+2n}}$, we have $0 < a_{n+1} < a_n$ and $\lim_{n\to\infty} a_n = 0$, and so by the Alternating Series Test, the power series converges at $x = -4$.

At $x = 10$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{6}{7^n\sqrt{n^2+2n}}(7)^n = \sum_{n=1}^{\infty} \frac{6}{\sqrt{n^2+2n}}.$$

Now $\frac{6}{\sqrt{n^2+2n}} \geq \frac{6}{\sqrt{3n^2}}$ for $n \geq 1$, and

$$\sum_{n=1}^{\infty} \frac{6}{\sqrt{3n^2}} = \sum_{n=1}^{\infty} \frac{6}{n\sqrt{3}}$$

diverges by the p-test (with $p = 1$). So by the (Direct) Comparison Test,

$$\sum_{n=1}^{\infty} \frac{6}{\sqrt{n^2+2n}}$$

diverges. Therefore, the interval of convergence for the power series is $[-4, 10)$.

Answer: $[-4, 10)$

7. [16 points] Gabriel the aspiring jazz musician owns a number of cats and kittens which he lets wander his neighborhood. When he wants to feed them, he blows his trusty cat trumpet, and waits for them to come running.

a. [6 points] The probability density function for the time t, in minutes, that it takes for Miles the cat to arrive is given by $m(t)$ where

$$m(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{5} & \text{for } 0 \le t \le a \\ \frac{1}{5}e^{-t+a} & \text{for } t > a \end{cases}$$

for some constant a. Find the value of a.

Solution: Since $m(t)$ is a pdf, we must have $\displaystyle\int_{-\infty}^{\infty} m(t)\ dt = 1$, and so

$$\begin{aligned} \frac{1}{5}a + \int_a^{\infty} \frac{1}{5}e^{-t+a}\ dt &= 1 \\ \frac{1}{5}a + \lim_{b\to\infty} \int_a^{b} \frac{1}{5}e^{-t+a}\ dt &= 1 \\ \frac{1}{5}a - \lim_{b\to\infty} \frac{1}{5}e^{-t+a}\Big|_a^b\ dt &= 1 \\ \tfrac{1}{5}a + \tfrac{1}{5} &= 1 \end{aligned}$$

and so $a = 5 - 1 = 4$.

b. [7 points] Find the mean time in minutes that it takes Miles to arrive. You should evaluate any integrals or limits in your expression. You may give your answer in terms of a, but not in terms of m. You are not required to simplify your answer.

Solution: The mean time is given by

$$\begin{aligned} \int_{-\infty}^{\infty} tm(t)\ dt &= \int_0^a \frac{t}{5}\ dt + \int_a^{\infty} \frac{t}{5}e^{-t+a}\ dt \\ &= \frac{a^2}{10} + \frac{1}{5}\lim_{b\to\infty} \int_a^b te^{-t+a}\ dt \\ &= \frac{a^2}{10} + \frac{1}{5}\lim_{b\to\infty} \left(-te^{-t+a}\Big|_a^b + \int_a^b e^{-t+a}\ dt \right) \\ &= \frac{a^2}{10} + \frac{1}{5}\lim_{b\to\infty} \left(\left(-te^{-t+a} - e^{-t+a}\right)\Big|_a^b \right) \\ &= \frac{a^2}{10} + \frac{1}{5}(a+1) \end{aligned}$$

c. [3 points] The cumulative distribution function for the amount of time that it takes for Ella the kitten to arrive is given by $E(t)$. Gabriel knows that 18% of the time Ella arrives in less than 2 minutes, and that 40% of the time she takes more than 6 minutes to arrive. Use this information to find the value of $E(6) - E(2)$.

Solution: From the given information, we see that $E(2) = 0.18$ and $E(6) = 1-0.4 = 0.6$. Therefore $E(6) - E(2) = 0.6 - 0.18 = 0.42$.

Answer: **0.42**

8. [14 points] Determine whether the following series converge or diverge. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

a. [7 points]

$$\sum_{n=1}^{\infty} \frac{3-\cos(n)}{6n-n^{1/2}}$$

Solution: We have, for $n \geq 1$,

$$\frac{3-\cos(n)}{6n-n^{1/2}} \geq \frac{2}{6n-n^{1/2}} \geq \frac{2}{6n} = \frac{1}{3n}$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n}$ diverges by the p-test (with $p = 1$).

Therefore, by the Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{3-\cos(n)}{6n-n^{1/2}}$ diverges.

b. [7 points]

$$\sum_{n=2}^{\infty} \frac{3}{n(\ln(n))^2}$$

Solution: Use the integral test, where $f(x) = \frac{3}{x(\ln(x))^2}$ is positive and decreasing.

$$\begin{aligned}\int_2^{\infty} \frac{3}{x(\ln(x))^2}dx &= \lim_{b\to\infty} \int_2^{b} \frac{3}{x(\ln(x))^2}dx \\ &= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{3}{u^2}du \quad \text{after using the substitution } u = \ln(x) \\ &= \int_{\ln 2}^{\infty} \frac{3}{u^2}du,\end{aligned}$$

which converges by the p-test with $p = 2$. Therefore, by the integral test, the original series converges as well.