Math 116 — Final Exam — December 10, 2020

1. **Do not write your name anywhere on this exam or your submissions.**
2. Your submissions must, however, have the correct problem number (if using blank paper) and your UMID and initials in the top corner of every page of every submission.
3. You must submit your work for each problem in Gradescope in the appropriate location.
4. Your submissions must be legible and rotated correctly.
5. This exam has 11 pages including this cover.
6. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
7. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
8. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
9. You may not communicate with any other individuals, message boards, forums, etc., either in person or virtually, while you are taking the midterm.
10. You may not use a calculator, phone, computer, tablet, laptop, or any other electronic device to assist you with any question.

    You are allowed any handwritten or printed notes, provided they are printed prior to your beginning this exam.
11. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
12. Include units in your answer where that is appropriate.
13. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	7	
2	15	
3	19	
4	16	
5	12	
6	11	

Problem	Points	Score
7	5	
8	6	
9	9	
Total	100	

1. [7 points] At a wildlife sanctuary, Diego fills the giraffes' water bowl at a constant rate of 0.5 gallons per minute. The rate in gallons per minute at which the giraffes drink from the bowl, t **minutes** after 8am, is given by $r(t)$. Suppose there are 12 gallons of water in the bowl at 10am.

a. [3 points] Write an expression possibly involving one or more integrals for the amount of water, in gallons, the giraffes drink between 9am and noon.

Solution: The rate that the giraffes drink water is given by $r(t)$, and so the total amount they drink from 9am to noon is given by

$$\int_{60}^{4(60)} r(t)\ dt = \int_{60}^{240} r(t)\ dt.$$

b. [4 points] Write an expression possibly involving one or more integrals for the amount of water, in gallons, in the bowl at 8am.

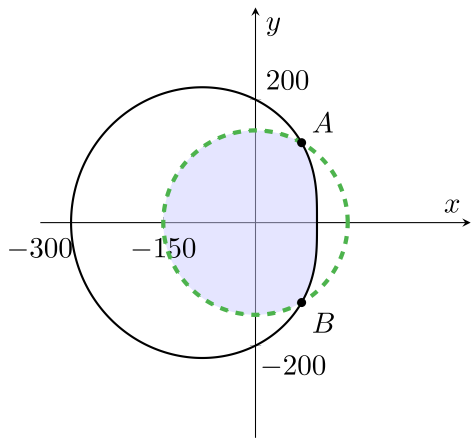
Solution: The volume of water in the bowl at 10am is 12 gallons, and so subtracting the change in volume between 8am and 10am will give the amount of water in the bowl at 8am.

The water is added to the bowl at a rate of 0.5 gallons per minute, and removed at a rate of $r(t)$ gallons per minute, and so the rate of change of volume of water in the bowl is $0.5 - r(t)$ gallons per minute. Therefore, the amount of water, in gallons, in the bowl at 8am is:

$$12 - \int_0^{2(60)} 0.5 - r(t)\ dt = 12 - \int_0^{120} 0.5 - r(t)\ dt = \int_0^{120} r(t)\ dt - 48.$$

2. [15 points] With a crashing stork market, the infinite trumpet glitch, and the forestry expansion over-expanding, the video game *Vegetable Crossing* has a lot of issues. Maria designs a new island for the game, and on the island there is an area where players can grow acacia plants.

- The island is in the shape of the polar curve $r = 200 - 100\cos(\theta)$ where $0 \le \theta < 2\pi$. The outline of the island is the **solid black curve** plotted below.
- **The acacia-growing zone is shaded blue**, and it is formed by the section of the island inside a circle of radius 150 meters centered at the origin. The circle is the dashed green curve plotted below.
- All distances on the graph are in meters.



a. [5 points] The points A and B, labeled above, are the intersection points of the polar curve $r = 200 - 100\cos(\theta)$ with the dashed green circle. Find points A and B **in polar coordinates**.

Solution: For the points of intersection, $200 - 100\cos(\theta) = 150$, and so $\cos(\theta) = \frac{1}{2}$. This means we must have $\theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$. Therefore, $A = (150, \frac{\pi}{3})$ and $B = (150, \frac{5\pi}{3})$.

b. [5 points] Find an expression involving one or more integrals for the length, in meters, of the perimeter of the acacia-growing zone. Do not evaluate your integral(s).

Solution: Part of the perimeter, is an arc of the circle of radius 150. The arc length of the section which is within the island is two-thirds $\left(\left(\frac{5\pi}{3} - \frac{\pi}{3}\right)/(2\pi)\right)$ of the circumference of the circle, and the circumference of the entire circle is 300π, so the arc length of this section is 200π.
For $f(\theta) = 200 - 100\cos(\theta)$, we have $f'(\theta) = 100\sin(\theta)$, and so the arc length of this section of the polar curve is given by

$$2\int_0^{\pi/3} \sqrt{(200 - 100\cos(\theta))^2 + (100\sin(\theta))^2}\, d\theta$$

where we have used the symmetry of the region.
In total then, the length of the perimeter is

$$200\pi + 2\int_0^{\pi/3} \sqrt{(200 - 100\cos(\theta))^2 + (100\sin(\theta))^2}\, d\theta.$$

c. [5 points] Players are able to pave any part of the island **outside** of the acacia-growing zone, at a cost of 7 dubloons per square meter. Find an expression involving one or more integrals for the cost, in dubloons, of paving the entire area which lies outside of the acacia-growing zone. Do not evaluate your integral(s).

Solution: The area outside the acacia-growing zone can be thought of as the area of the sector of the island with $\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$ with the area inside the circle in this sector subtracted. The area of this sector of the circle is again two-thirds of the area of the circle, which is equal to $\frac{2}{3}\pi(150)^2$, and so the total area of the acacia-growing zone is

$$\frac{1}{2}\int_{\pi/3}^{5\pi/3} (200 - 100\cos(\theta))^2\, d\theta - \frac{2}{3}\pi(150)^2.$$

This means that the cost, in dubloons, is

$$7\left(\frac{1}{2}\int_{\pi/3}^{5\pi/3} (200 - 100\cos(\theta))^2\, d\theta - \frac{2}{3}\pi(150)^2\right).$$

3. [19 points] Consider the function $B(x)$ described on its domain by its Taylor series around $x = 0$,

$$B(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!(2n)!} x^{2n}.$$

a. [5 points] Find the first four non-zero terms of the Taylor series for $B(x)$ about $x = 0$. You do not need to evaluate any factorials in your answer.

Solution:

$$B(x) = 1 - \frac{1}{2!2!}x^2 + \frac{1}{3!4!}x^4 - \frac{1}{4!6!}x^6 + \ldots$$

b. [6 points] Find the radius of convergence of the Taylor series. Show all of your work and use proper notation.

Solution: We use the ratio test with $a_n = \frac{(-1)^n}{(n+1)!(2n)!}x^{2n}$.
We have:

$$\begin{aligned}\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{(n+1)!(2n)!|x^{2n+2}|}{(n+2)!(2n+2)!|x^{2n}|} \\ &= \lim_{n\to\infty} \frac{x^2}{(n+2)(2n+2)(2n+1)} \\ &= 0 \text{ for all } x.\end{aligned}$$

Therefore, the radius of convergence is ∞.

c. [3 points] Is $B(x)$ an odd function, an even function, or neither? Explain your reasoning.

Solution: The Taylor series contains only even powers of x, and so $B(x)$ is an even function.

d. [5 points] Find the value of $B^{(2020)}(0)$. You do not need to evaluate any factorials in your answer.

Solution: We know that $\frac{B^{(2020)}(0)}{2020!}x^{2020}$ appears in the Taylor series for $B(x)$. Comparing this with the given expression for $B(x)$, we see we must have $x^{2n} = x^{2020}$, i.e. $n = 1010$. Therefore, comparing coefficients of x^{2020},

$$\frac{B^{(2020)}(0)}{2020!} = \frac{(-1)^{1010}}{1011!2020!},$$

and so

$$B^{(2020)}(0) = \frac{1}{1011!}.$$

4. [16 points] Buluu and Nyekundu the giraffes work on another art project. They paint on a large, square canvas, with corners at $(0, 0)$, $(0, 100)$, $(100, 0)$, and $(100, 100)$, with distances in yards. As Buluu moves, he leaves a trail of blue paint. As Nyekundu moves, she leaves a trail of red paint.

Buluu's position t minutes after the giraffes start painting is described by the parametric equations:

$$\begin{cases} x(t) = t^2 \\ y(t) = 40 + 2(t-5)^2 \end{cases}$$

and Nyekundu's position t minutes after the giraffes start painting is described by the parametric equations:

$$\begin{cases} x(t) = 25 + 20\sin(\pi t) \\ y(t) = 40 - (t-5)^2 \end{cases}$$

a. [5 points] If the giraffes ever collide, they produce a purple splotch on the canvas at the point of collision. Will the giraffes produce any purple splotches? If so, at **what time(s)** does this occur, and **where on the canvas** are the purple splotches located?

Solution: Notice that for Buluu, $y(t) \geq 40$, and for Nyekundu, $y(t) \leq 40$, so their y-coordinates are only equal when $y(t) = 40$. This happens for both giraffes when $t = 5$. At this time, both have $x(t) = 25$, and so there is a purple splotch created at $t = 5$ at $(25, 40)$.

b. [6 points] Find an expression for Buluu's speed 3 minutes after the giraffes start painting. Make sure to include units.

Solution: For Buluu, $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 4(t-5)$, and so Buluu's speed at $t = 3$ is $\sqrt{(2(3))^2 + (4(-2))^2} = \sqrt{36+64} = \sqrt{100} = 10$ yards per minute

c. [5 points] At what time does Nyekundu first leave the canvas? Make sure to fully justify your answer.

Solution: For Nyekundu, $5 \leq x(t) \leq 45$, and so her x-coordinate is never less than 0 or bigger than 100.

She also has $y(t) \leq 40$, and so she leaves the canvas when $y(t) = 0$, i.e when $(t-5)^2 = 40$. Solving gives $t = 5 + \sqrt{40}$ as the only (positive) time when this happens. Since $y(t)$ is negative for all larger values of t, Nyekundu does leave the canvas at this time.

5. [12 points] For each of the following, write down the CAPITAL LETTER for the **ONE** best answer for your submission. You do not need to show work, but it is strongly suggested that you try to solve the problem from start to finish before selecting your answer.

a. [4 points] A container in the shape of a cube with side length ℓ meters is resting with one face on the ground. It is filled 1/3 of the way with a liquid of constant density 2000 kg/m^3. Recall the gravitational constant is $g = 9.8\text{m/s}^2$. How much work in joules is required to pump all the liquid to the top of the container?

(**A**) $\displaystyle\int_0^{\frac{1}{3}\ell} 2000g(\ell - h)h^2\, d\ell$

(**B**) $\displaystyle\int_0^{\frac{1}{3}\ell} 2000g(\ell - h)\ell^2\, dh$

(**C**) $\displaystyle\int_{\frac{1}{3}\ell}^{\ell} 2000g(\ell - h)h^2\, dh$

(**D**) $\displaystyle\int_{\frac{1}{3}\ell}^{\ell} 2000g(\ell - h)\ell^2\, d\ell$

(**E**) None of the above.

b. [4 points] Suppose $p(x)$ is a probability density function for x, the total weight of avocados, **in millions of pounds**, eaten in the United States on a given day. The equation $p(1) = 0.5$ can be best interpreted as: "On any given day,...

(**A**)...there is a 50% chance that 1 million pounds of avocados will be eaten in the United States."

(**B**) ...there is roughly 100% chance that between 0 and 2 million pounds of avocados will be eaten in the United States."

(**C**) ...there is approximately a 10% chance that between 900,000 and 1,100,000 pounds of avocados will be eaten in the United States."

(**D**) ...the median weight of avocados eaten in the United States is 1 million pounds."

(**E**) None of the above.

c. [4 points] Consider the series

$$\sum_{n=1}^{\infty} a_n = 0 - \frac{1}{2} + 0 - \frac{1}{4} + 0 - \frac{1}{6} + 0 - \frac{1}{8} + ...$$

which is the infinite sum of the terms in the sequence defined by

$$a_n = \begin{cases} 0 & \text{for } n \text{ odd.} \\ -\frac{1}{n} & \text{for } n \text{ even.} \end{cases}$$

The series $\displaystyle\sum_{n=1}^{\infty} a_n$...

(**A**) ...converges by the alternating series test.

(**B**) ...converges because it converges absolutely.

(**C**) ...converges by the ratio test.

(**D**) ...converges because $\displaystyle\lim_{n\to\infty} a_n = 0$.

(**E**) None of the above.

6. [11 points]

a. [6 points] Find the Taylor series about $x = 0$ for the function $f(x) = 3 + \cos(2x^2)$. Write your answer using sigma notation and also write out the first **three** non-zero terms. You do not need to simplify any factorials or exponentials that appear in your answer.

Solution: We have $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, and so,

$$\cos(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{2n}}{(2n)!}.$$

So

$$3 + \cos(2x^2) = 3 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{2n}}{(2n)!} = 4 - \frac{(2x^2)^2}{2!} + \frac{(2x^2)^4}{4!} - \dots$$
$$= 4 - \frac{2^2 x^4}{2!} + \frac{2^4 x^8}{4!} - \dots$$

b. [5 points] The function $f(x)$ from part a) has an antiderivative $F(x)$ which satisfies $F(0) = 9$. Find the first four nonzero terms of the Taylor series about $x = 0$ for $F(x)$. You do not need to simplify any factorials or exponentials that appear in your answer.

Solution: Integrating term by term, and using $F(0) = 9$, we see

$$F(x) = 9 + 4x - \frac{2^2 x^5}{5(2!)} + \frac{2^4 x^9}{9(4!)} - \dots$$

7. [5 points] Find an expression for the exact value of

$$12 + \frac{4}{5} - \frac{4^2}{2(5)^2} + \frac{4^3}{3(5)^3} + \dots + \frac{(-1)^{n+1} 4^n}{n 5^n} + \dots$$

which does not involve an infinite sum (i.e. no sigma notation or "...").

Solution: Using our known Taylor series, we see that this is $12 + \ln\left(1 + \frac{4}{5}\right) = 12 + \ln\left(\frac{9}{5}\right)$.

8. [6 points] Suppose

$$F(x) = \int_3^{2x} (e^{5t^2} - 2)\ dt.$$

Find all x-values where the graph of $y = F(x)$ has a horizontal tangent line, showing all of your work. You do not need to simplify your answer(s).

Solution: The graph has a horizontal tangent line where $F'(x) = 0$.
Using the Second Fundamental Theorem, we see $F'(x) = 2(e^{5(2x)^2} - 2) = 2(e^{20x^2} - 2)$, and so $F'(x) = 0$ when $e^{20x^2} = 2$, i.e. $20x^2 = \ln 2$.
Therefore, $x = \pm\sqrt{\frac{\ln 2}{20}}$.

9. [9 points] For each part of this problem, write the CAPITAL LETTER corresponding to **ALL** answers that apply on your submission. You do not need to show your work.

a. [5 points] Suppose $f(x)$ is a continuous function defined for $x \geq 1$ satisfying:

- $f(x) > 0$ for all $x \geq 1$.
- $f(x)$ is decreasing on its domain.
- $f(x) \leq \frac{1}{\sqrt{x}}$

Which of the following MUST be true about $f(x)$?

(**A**) $\int_1^{\infty} f(x)\ dx$ converges.

(**B**) $\int_1^{\infty} (f(x))^2\ dx$ converges.

(**C**) $\int_1^{\infty} \frac{f(x)}{x}\ dx$ converges. [boxed]

(**D**) $\int_1^{\infty} f(x)\ dx$ diverges.

(**E**) None of the above.

b. [4 points] Which of the following pairs of polar coordinates are the same point in the xy-plane as the point $(x, y) = (-1, 1)$?

(**A**) $(r, \theta) = \left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$

(**B**) $(r, \theta) = \left(1, \frac{3\pi}{4}\right)$

(**C**) $(r, \theta) = \left(-\sqrt{2}, -\frac{\pi}{4}\right)$ [boxed]

(**D**) $(r, \theta) = \left(-\frac{\sqrt{2}}{2}, \frac{5\pi}{4}\right)$

(**E**) None of the above.

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin\theta$	$\cos\theta$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$