Math 116 — First Midterm — October 4, 2021

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1. **Do not write your name anywhere on this exam or your submissions.**
2. Your submissions must be written on blank paper, problems must be numbered correctly, and your UMID and initials in the top corner of every page of every submission.
3. You must submit your work for each problem in Gradescope in the appropriate location.
4. Your submissions must be legible and rotated correctly.
5. This exam has 12 pages including this cover.
6. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
7. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
8. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
9. You may not communicate with any other individuals, message boards, forums, etc., either in person or virtually, while you are taking the midterm.
10. You may not use a calculator, phone, computer, tablet, laptop, or any other electronic device to assist you with any question.

    You may not use any other resources (e.g. book, notes, etc..), except you are allowed one two-sided 3"x5" notecard for this exam.
11. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
12. Include units in your answer where that is appropriate.
13. Problems may ask for answers in *exact form.* Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	15	
2	14	
3	13	
4	16	
5	11	
6	13	

Problem	Points	Score
7	8	
8	10	
Total	100	

1. [15 points] Below are a table of values for a function $f(x)$ which is **odd** and twice differentiable.

x	0	1	2	3	4
$f(x)$	0	2	-1	4	1
$f'(x)$	1	5	e	2	0

Use the table to compute the following quantities. Show your work.

a. [4 points] Approximate the integral $\displaystyle\int_{-1}^{1} f(2x+2)dx$ using MID(2). Write out each term in your sum.

Solution: Set $g(x) = f(2x+2)$. Then,

$$\text{MID}(2) = 1\big(g(-\frac{1}{2}) + g(\frac{1}{2})\big) = f(1) + f(3) = 2 + 4 = 6.$$

b. [4 points] $\displaystyle\int_{-3}^{3} f'(x)(2x+2)dx$.

Solution: Using integration by parts($u = 2x+2$ and $dv = f'(x)dx$) and the fact that f is odd:

$$\begin{aligned}\int_{-3}^{3} f'(x)(2x+2)dx &= (2x+2)f(x)\Big|_{-3}^{3} - 2\int_{-3}^{3} f(x)dx \\ &= 8f(3) - (-4)(f(-3)) \\ &= 8(4) - (-4)(-4) = 16.\end{aligned}$$

Note that since f is odd, $\displaystyle\int_{-3}^{3} f(x)dx = 0$.

c. [3 points] $\displaystyle\int_{-1}^{1} (x+1)f'((x+1)^2)dx$.

Solution: Substitute $w = (x+1)^2$ to obtain

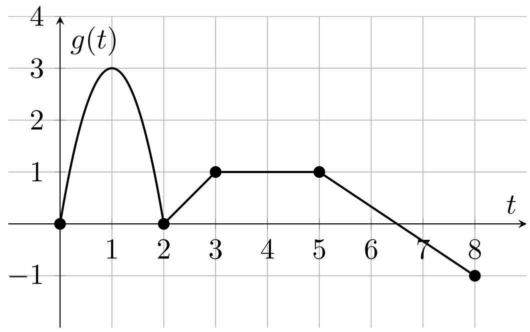
$$\int_{-3}^{3} (x+1)f'((x+1)^2)dx = \frac{1}{2}\int_{0}^{4} f'(w)dw = \frac{1}{2}f(w)\Big|_{0}^{4} = \frac{1}{2}(f(4) - f(0)) = \frac{1}{2}(1-0) = \frac{1}{2}.$$

d. [4 points] The average value of $(f(x)+1)^2 f'(x)$ on $[2,4]$.

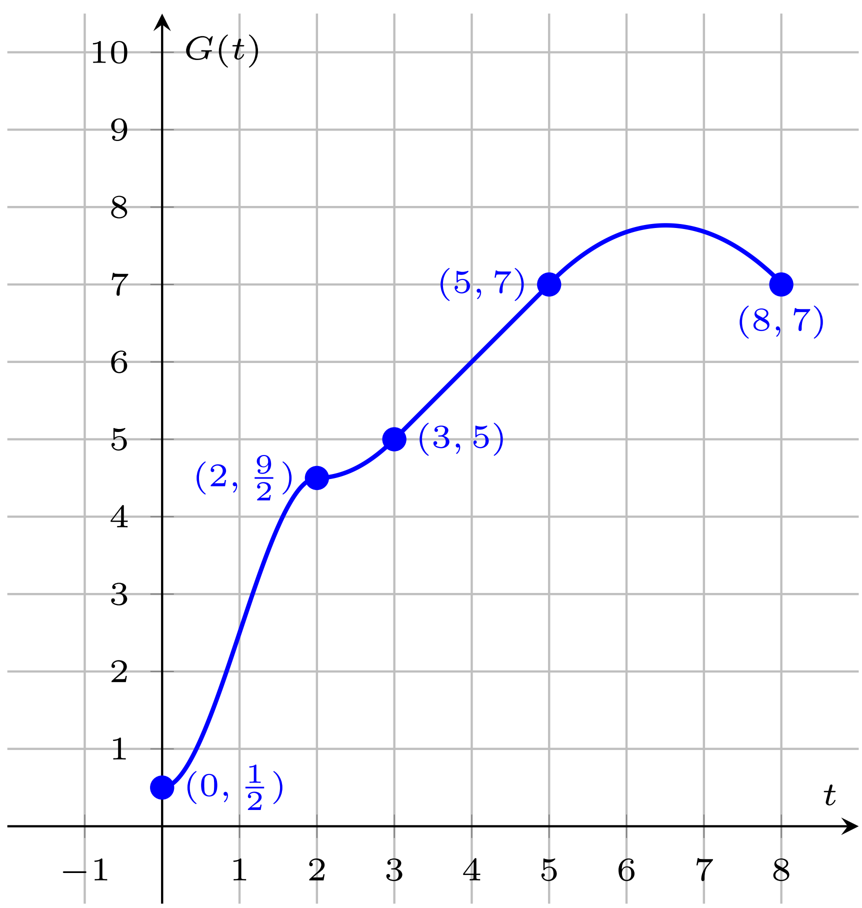
Solution: Substitute $w = f(x)$ to obtain

$$\frac{1}{2}\int_{2}^{4} (f(x)+1)^2 f'(x)dx = \frac{1}{2}\int_{f(2)}^{f(4)} (w+1)^2 dw = \frac{(w+1)^3}{6}\Big|_{-1}^{1} = \frac{1}{6}(8-0) = \frac{4}{3}.$$

2. [14 points] Shirley is trying to measure the right amount of sugar into a bowl for 8 seconds. The function $g(t)$ gives the rate (in cups/second) at which the amount of sugar in the bowl is changing t seconds after she starts measuring. The graph is linear on the intervals $[2, 3]$, $[3, 5]$, $[5, 8]$, and quadratic on $[0, 2]$ with formula $g(t) = 6t - 3t^2$:



Sketch a detailed graph of $G(t)$, the antiderivative of $g(t)$, giving the amount of sugar in the bowl at time t assuming there are 5 cups of sugar in the bowl after 3 seconds. Only graph $G(t)$ on the interval [0,8]. Make sure to clearly label the output and input of the points at $t = 0, 2, 3, 5, 8$. Be sure to make it clear where the graph is concave up, concave down, or linear and where it is increasing or decreasing. Use **hand-drawn** axes similar to those given below.



3. [13 points] Miley and Kylie see their friend Brian in the distance and decide to race to see who can reach him first. However, they see Brian begin pacing back and forth so depending on when they start the race, they will run a different amount. The distance they run, in meters, if the race starts t seconds after Brian begins pacing is

$$L(t) = 25 + 4 \int_{-\left(\frac{\pi}{2}\right)^{\frac{1}{3}}}^{t^3} \cos(r^3) r^2 dr$$

Throughout this problem, please give answers in **exact form** and include **units.**

a. [4 points] If Miley and Kylie start the race immediately as Brian begins pacing, what distance will they run? Evaluate any integrals in your answer and remember to include units.

Solution: The initial value is $L(0)$. Substituting $w = r^3$,

$$\begin{aligned} L(0) &= 25 + 4 \int_{-(\frac{\pi}{2})^{\frac{1}{3}}}^{0} \cos(r^3) r^2 dr \\ &= 25 + \frac{4}{3} \int_{-\frac{\pi}{2}}^{0} \cos w dw = 25 + \frac{4}{3} \left(\sin w \Big|_{-\frac{\pi}{2}}^{0} \right) \\ &= 25 + \frac{4}{3}(0 - (-1)) = 25 + \frac{4}{3} \text{ meters.} \end{aligned}$$

b. [6 points] Miley and Kylie decide they will start the race at the smallest strictly positive time t (i.e. smallest t with $t > 0$) such that $L'(t) = 0$. Find the time at which they will start the race. Make sure to include units.

Solution: Taking the derivative using the chain rule,

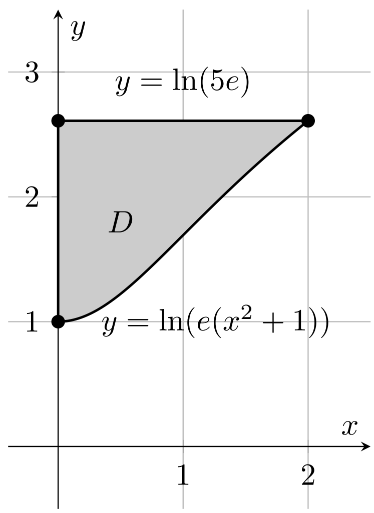
$$L'(t) = 4 \cos((t^3)^3)(t^3)^2 3t^2 = 12t^8 \cos(t^9).$$

Since $t^8 \geq 0$ for $t \geq 0$, the smallest strictly positive time for which $L'(t) = 0$ occurs at the smallest positive time for which the cosine term is 0. This happens when $t^9 = \frac{\pi}{2}$. Solving this gives that the race will start at $t = \left(\frac{\pi}{2}\right)^{\frac{1}{9}}$ seconds.

c. [3 points] Miley and Kylie want to be able to compute $L(t)$ quickly, so they would like $L(t)$ rewritten in the form below. Write $L(t)$ in the form given below with appropriate expressions in place of the blanks.

$$L(t) = \underline{\qquad 25 + \tfrac{4}{3} \qquad} + \int_0^t \underline{\qquad 12r^8 \cos(r^9) \qquad}\, dr$$

4. [16 points] Consider the region D in the $xy-$plane bounded between $y = \ln(e(x^2+1))$ and $y = \ln(5e)$ for x values between 0 and 2. A sketch of the region is shown below.



a. [6 points] Using the washer method, find an expression involving one or more integrals for the volume of the solid formed by rotating the region D around the x-axis. Do not evaluate your integral(s).

Solution: Taking vertical slices, we see that we obtain the washer method. The volume of a slice of thickness Δx at horizontal coordinate x is approximately

$$\Delta V = \pi((\ln(5e))^2 - (ln(e(x^2+1)))^2)\Delta x$$

and so the total volume of the solid is

$$\int_0^2 \pi((\ln(5e))^2 - (ln(e(x^2+1)))^2)dx.$$

b. [6 points] Using the shell method, find an expression involving one or more integrals for the volume of the solid formed by rotating the region D around the line $x = 2$. Do not evaluate your integral(s).

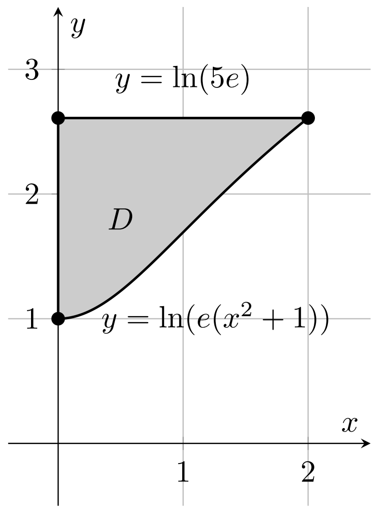
Solution: Taking vertical slices, we see that we obtain the shell method. Since we are rotating around $x = 2$, the horizontal distance between x and the axis of rotation is $2-x$ for any $x \in [0, 2]$. The volume of a slice of thickness Δx at horizontal coordinate x is approximately

$$\Delta V = 2\pi(2-x)(\ln(5e) - \ln(e(x^2+1)))\Delta x$$

and so the total volume of the solid is

$$\int_0^2 2\pi(2-x)(\ln(5e) - \ln(e(x^2+1)))dx.$$

4. (continued) Here is a reproduction of the plot on the previous page:



c. [4 points] Find an expression involving one or more integrals for the perimeter of the region D. Do not evaluate your integral(s).

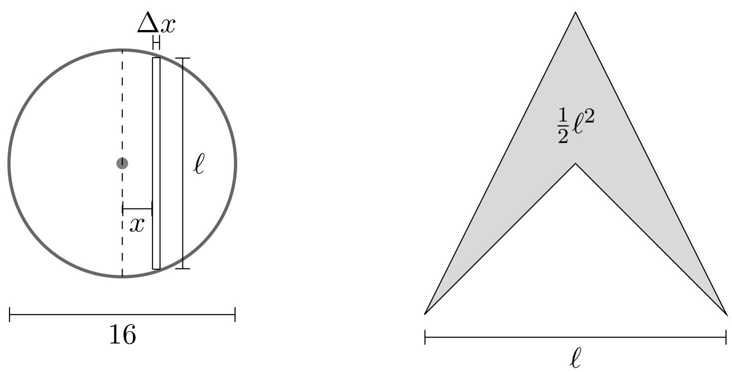
Solution: The boundary of the region D has 3 sides. The left side of D has length $\ln(5e) - 1 = \ln(5)$. The top side of D has length 2. To calculate the length of the lower curve, set $f(x) = \ln(e(x^2 + 1))$. Then

$$f'(x) = \frac{2x}{x^2 + 1}.$$

Using the arc length formula, we get that the perimeter is

$$\ln(5) + 2 + \int_0^2 \sqrt{1 + \left(\frac{2x}{x^2 + 1}\right)^2}\,dx.$$

5. [11 points] Tony's climbing gym wants to put in a climbing structure based off of the Chicago Bean. However, they want to make it more angular. The base of the structure will be in the shape of a circle with an 8 meter radius. The cross-sections perpendicular to the circle lying above a slice of the circle of length ℓ meters (as shown below) have area $\frac{1}{2}\ell^2$ square meters and are pictured below. The density of the material used to build the structure is not constant and has density dependent on its horizontal distance x from the vertical diameter through the circle. The density in kg/m^3 is given by $\delta(x) = 1000\sqrt{1 + x^2}$.



a. [2 points] Write an expression that gives the quantity ℓ in terms of x.

Solution: Using x and ℓ as defined in the figure above, we have $x^2 + \left(\frac{\ell}{2}\right) = 8^2$. Solving this, since $\ell \geq 0$,

$$\ell = 2\sqrt{64 - x^2}.$$

b. [3 points] Write an expression that gives the approximate volume, in cubic meters, of a slice of the structure a horizontal distance x meters away from the diameter of the circle with thickness Δx. Your expression should not involve any integrals.

Solution: The approximate volume of a slice of the structure x meters away from the vertical diameter of the circle, using (a) is

$$\frac{1}{2}\ell^2\Delta x = 2(64 - x^2)\Delta x.$$

c. [3 points] Using your expression from (b) to write an expression involving integrals which gives the total volume of the structure in cubic meters. Do not evaluate any integrals.

Solution: The volume is the integral of the volume of the slice in (b) as $\Delta x \to 0$, from $x = -8$ to $x = 8$ (or alternatively twice the integral from $x = 0$ to $x = 8$):

$$2\int_0^8 2(64 - x^2)dx = 4\int_0^8 64 - x^2 dx.$$

d. [3 points] Write an expression involving integrals which gives the total mass of the structure in kg. Your answer may contain $\delta(x)$. Do not evaluate any integrals.

Solution: Using our answer from (b) and the density of the slice a horizontal distance x meters away from the vertical diameter of the circle, the approximate mass of a slice of the structure is

$$2\delta(x)(64 - x^2)\Delta x.$$

The mass is the integral of the mass of the slice as $\Delta x \to 0$ from $x = -8$ to $x = 8$ (or alternatively twice the integral from $x = 0$ to $x = 8$ since $\delta(x)$ is even):

$$4\int_0^8 \delta(x)(64 - x^2)dx.$$

6. [13 points]

a. [6 points] Split the function $\frac{4-9x}{(x-2)^2(x+5)}$ into partial fractions with 2 or more terms. **Do not integrate these terms**. Please show all work to obtain your partial fractions.

Solution: Let

$$\frac{4-9x}{(x-2)^2(x+5)} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x+5)}.$$

Then,

$$A(x-2)(x+5) + B(x+5) + C(x-2)^2 = 4 - 9x.$$

Comparing coefficients of x^2

$$A + C = 0 \text{ , so } A = -C.$$

Comparing coefficients of x,

$$3A + B - 4C = -9 \text{ , so } B = -9 - 7A.$$

Comparing the coefficients of the constant term,

$$4C + 5B - 10A = 4 \text{ , so } A = -1.$$

Substituting back, we see that $B = -2$ and $C = 1$, so

$$\frac{4-9x}{(x-2)^2(x+5)} = \frac{-1}{(x-2)} + \frac{-2}{(x-2)^2} + \frac{1}{(x+5)}.$$

b. [7 points] Use the fact that $\frac{5x}{(x^2+1)(x-2)} = \frac{2}{x-2} + \frac{-2x+1}{(x^2+1)}$ to solve the indefinite integral

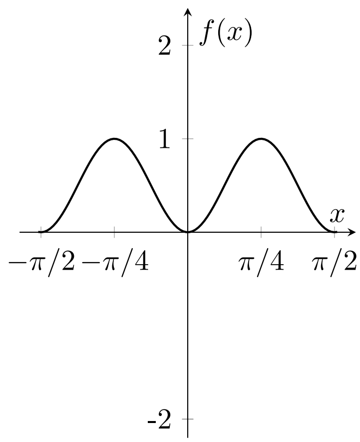
$$\int \frac{5x}{(x^2+1)(x-2)} dx.$$

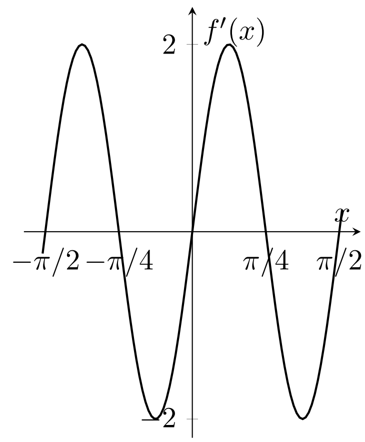
Solution: We first split the integral into 3 terms and use substitution $w = x^2 + 1$ on the second term:

$$\begin{aligned}
\frac{5x}{(x^2+1)(x-2)dx} &= \int \frac{2}{x-2} dx + \int \frac{-2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx \\
&= \int \frac{2}{x-2} dx - \int \frac{1}{w} dw + \int \frac{1}{x^2+1} dx \\
&= 2\ln|x-2| - \ln|w| + \arctan x + C \\
&= 2\ln|x-2| - \ln(x^2+1) + \arctan x + C.
\end{aligned}$$

Note that we have substituted back in for x.

7. [8 points] Consider the function $f(x) = \sin^2(2x)$. The graph of f and f' are shown below.





Determine whether the following approximations of integrals of $f(x)$ are overestimates or underestimates. Clearly write the entire word, either OVERESTIMATE or UNDERESTIMATE. If it cannot be determined whether the estimate is an over- or underestimate using the methods of the course, write CANNOT DETERMINE. You do not need to show your work.

a. [2 points] LEFT(4) of $\displaystyle\int_0^{\pi/4} f(x)dx$.

OVERESTIMATE $\boxed{\text{UNDERESTIMATE}}$ CANNOT DETERMINE

b. [2 points] RIGHT(4) of $\displaystyle\int_{\pi/4}^{\pi/3} f(x)dx$.

OVERESTIMATE $\boxed{\text{UNDERESTIMATE}}$ CANNOT DETERMINE

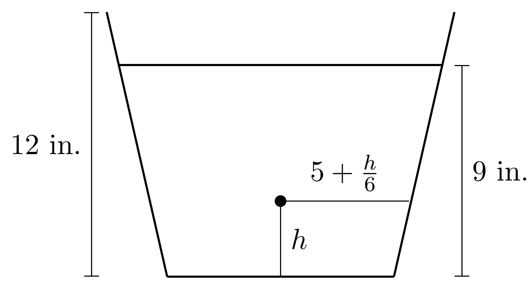
c. [2 points] TRAP(4) of $\displaystyle\int_{-\pi/8}^{\pi/8} f(x)dx$.

$\boxed{\text{OVERESTIMATE}}$ UNDERESTIMATE CANNOT DETERMINE

d. [2 points] MID(4) of $\displaystyle\int_0^{\pi/12} f(x)dx$.

OVERESTIMATE $\boxed{\text{UNDERESTIMATE}}$ CANNOT DETERMINE

8. [10 points] Frank, an aspiring chef and looking to impress his extended family, makes a big pot of tomato soup that he brings to his family reunion potluck. The pot is 12 inches tall with **circular cross sections** parallel to the bottom of the pot. The cross section h inches from the bottom of the pot has radius $5 + \frac{h}{6}$ inches for $0 \le h \le 12$. Unfortunately, his least favorite cousin Barrie brought a better tomato soup to the potluck. Almost no one ate Frank's soup and after the potluck, the pot still had soup up to 9 inches above the bottom of the pot. Frank saved the rest of the soup for himself and now he has to get the remaining soup out of the pot.



a. [3 points] Write an expression for the volume of a thin horizontal slice of soup at height h from the bottom of the pot and thickness Δh. Make sure to include units.

Solution: Taking our horizontal slice the volume ΔV can be expressed using the fact that the radius $r = 5 + \frac{h}{6}$:

$$\Delta V = \pi r^2 \Delta h = \pi(5 + \frac{h}{6})^2 \Delta h \text{ in.}^3$$

b. [4 points] By the end of the potluck, the soup has settled into an uneven density. The density of the soup at height h above the bottom of the pot is $.05(1+h)$ pounds/in^3. Write an expression for the amount of work in pound-inches required to get a thin horizontal slice of soup at height h above the bottom of the pot and thickness Δh to the top of the pot.

Solution: The thin horizontal slice at height h above the bottom of the pot and thickness Δh must be raised $12 - h$ inches. It has density $.05(1 + h)$ pounds/in^3. So, the work ΔW done in raising the horizontal slice is

$$\Delta W = .05(1 + h)\pi(5 + \frac{h}{6})^2(12 - h)\Delta h.$$

c. [3 points] Write a definite integral that represents the total amount of work in pound-inches required to get all the soup that was left in the pot after the potluck out of the pot. Do not evaluate the integral.

Solution: The pot is 9 in. full, so the total work done in raising the soup is

$$\int_0^9 .05\pi(1 + h)(5 + \frac{h}{6})^2(12 - h)dh.$$