Math 116 — Second Midterm — November 8, 2021

Write your 8-digit UMID number very clearly in the box to the right.

Your Initials Only: ______ Instructor Name: ________________________ Section #: ___

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided. **You will receive 4 points on this exam for doing this correctly.**
2. This exam has 10 pages including this cover.
3. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	4	
2	9	
3	14	
4	12	
5	10	
6	14	

Problem	Points	Score
7	8	
8	9	
9	20	
Total	100	

1. [4 points] If you have not already done so, neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.

2. [9 points] The chain grocery store Mathmart stocks their shelves full of brain-targeted foods for their customers on a regular basis. The time (in minutes) it takes to stock the shelves after the delivery truck arrives is represented by the cumulative distribution function (cdf):

$$R(t) = \begin{cases} 0 & t < 1 \\ 1 - \frac{1}{t^3} & t \geq 1 \end{cases}$$

a. [3 points] Find a formula for $r(t)$, a probability density function (pdf) corresponding to $R(t)$.

Solution: We must have $r(t) = R'(t)$ whenever it is defined. Using our formula for part (a), we see that a possible formula is

$$\begin{cases} 0 & t < 1 \\ 3\frac{1}{t^4} & t \geq 1, \end{cases}$$

where we have chosen $r(1) = 3$. We could have chosen $r(1)$ to be any non-negative number since $R'(t)$ is not defined at 1.

b. [3 points] What is the median amount of time (in minutes) it takes to stock the shelves?

Solution: The median occurs when $R(t) = .5$. Setting

$$1 - \frac{1}{t^3} = .5$$
$$.5 = \frac{1}{t^3}$$
$$t = 2^{\frac{1}{3}}.$$

Therefore, the median amount of time to stock the shelves in minutes is $2^{\frac{1}{3}}$ minutes.

c. [3 points] Write an integral representing the mean amount of time (in minutes) it takes to stock the shelves. Your expression should not contain r or R. Do not solve this integral.

Solution:

$$\int_1^{\infty} \frac{3}{t^3} dt$$

3. [14 points] Molly has recently become a sheep herder. She rotates her sheep through various fields so that the sheep have a varied diet and the fields have a chance to grow. Every Monday, the sheep visit the same field. Before the sheep graze for the first time in this field, its grass is 20 centimeters tall. Molly's sheep are picky and only eat the top 40% of the length of grass in this field every Monday. Over the course of the week, before the next Monday, the grass grows 3 centimeters. Let G_i represent the height in centimeters of the grass right before the sheep graze on it for the ith time. Note that $G_1 = 20$.

a. [5 points] Find expressions for each of G_2, G_3, and G_4. You do not need to evaluate your expressions.

Solution:

$$\begin{aligned} G_2 &= (0.6)G_1 + 3 \\ &= (0.6)(20) + 3 \\ G_3 &= (0.6)G_2 + 3 \\ &= (0.6)^2(20) + (0.6)(3) + 3 \\ G_4 &= (0.6)G_3 + 3 \\ &= (0.6)^3(20) + (0.6)^2(3) + (0.6)(3) + 3 \end{aligned}$$

b. [5 points] Find a general **closed-form** expression for G_n, defined for $n = 2, 3, 4 \ldots$

Solution:

$$\begin{aligned} G_n &= (0.6)^{n-1}(20) + \sum_{i=0}^{n-2} 3(0.6)^i \\ &= (0.6)^{n-1}(20) + \frac{3(1-(0.6)^{n-1})}{1-0.6} \end{aligned}$$

c. [4 points] In order for the field to meet sheep grazing standards, the height of the grass must be at least 5 cm when the sheep begin grazing. Molly thinks she will be able to stay on her field forever. Help her determine whether she can stay by either showing that the grass will eventually be less than 5 cm in height, or showing that the grass will be at least 5 cm each time before the sheep graze.

Solution:

$$\lim_{n\to\infty} G_n = \frac{3}{1-0.6} = 7.5.$$

Also note that G_n is a decreasing sequence. So, the grass is always taller than 5 cm. when the sheep begin grazing.

4. [12 points]

a. [6 points] Determine if the following integral converges or diverges and circle the corresponding word. You must show all work and indicate any theorems you use. You do not need to calculate the value of the integral if it converges:

$$\int_4^\infty \frac{x}{3x^2 - \ln x}dx$$

Circle one: **Converges** **Diverges**

Solution: On the interval $[4, \infty)$,

$$\frac{x}{3x^2 - \ln x} \geq \frac{x}{3x^2} = \frac{1}{3x}.$$

By the p-test ($p = 1$), $\int_4^\infty \frac{1}{3x}dx$ diverges. Therefore, by the (Direct) Comparison Test, $\int_4^\infty \frac{x}{3x^2 - \ln x}dx$ diverges.

b. [6 points] Determine whether the following series converges or diverges and circle the appropriate answer. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use:

$$\sum_{n=1}^\infty \frac{100^n}{n(99^n)}$$

Circle one: **Converges** **Diverges**

Solution: Let $a_n = \frac{100^n}{n(99)^n}$.

Method 1:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(100/99)^n}{n} \overset{\text{L'H}\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{\ln(100/99)(100/99)^n}{1} = \infty.$$

So, by the nth term test, the series $\sum_{n=1}^\infty a_n$ diverges.

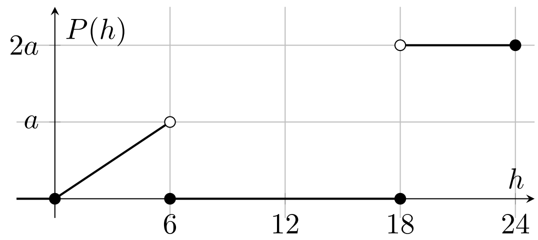
Method 2: Using the ratio test, since $a_n \geq 0$,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{100^{n+1}}{(n+1)(99)^{n+1}} \frac{n(99)^n}{100^n} = \lim_{n\to\infty} \frac{100n}{99(n+1)} = \frac{100}{99} > 1.$$

So, by the Ratio Test, the series $\sum_{n=1}^\infty a_n$ diverges.

5. [10 points] George's mom's birthday party is in 24 hours and George still hasn't bought her a present. The mall near George is open for the next 6 hours, then closes for 12 hours, and then is open another 6 hours tomorrow before the party starts. George will search the mall until he finds the perfect present.

Below is a **partial** graph of $P(h)$, the probability density function (pdf) representing how long it will take George to find the perfect present in h hours. Assume $a > 0$ is some constant and $P(h) = 0$ for any $h \leq 0$.



a. [3 points] If the probability George finds the perfect present for his mom before the party starts is 1, what is the correct value of a in the graph above?

Solution: The area under the part of the graph that is visible is $\frac{1}{2}(6a) + 12a$. Setting this to be 1,

$$15a = 1$$
$$a = \frac{1}{15}$$

It is important to note for parts ***b.*** *and* ***c.*** *that only a* ***partial*** *graph of the function $P(h)$ is shown.*

b. [3 points] Now suppose $a = \frac{1}{20}$. What is the probability George will **not** find a present before the start of the party?

Solution: Now the area under the graph is still $15a$, but now $15a = \frac{15}{20} = \frac{3}{4}$. So, there is a 25% chance George will not find a present.

c. [4 points] In the case that $a = \frac{1}{20}$, finish the sentence to write a practical interpretation for the statement $P(26) = .02$:

There is approximately a 1% chance that....

Solution: There is approximately a 1% chance that George will take between 26 & 26.5 hours to find a present for his mom.
Note that $(26.5 - 26).02 = .01 = 1\%$. In particular, we could have used any interval around 26 that has length 0.5.

6. [14 points] Determine whether each of the following series converge conditionally, converges absolutely, or diverges and circle the appropriate answer. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

a. [7 points]

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{6+\sqrt{n}}$$

Circle one: **Absolutely Converges** **Conditionally Converges** **Diverges**

Solution: First, we use the alternating series test to show the series itself converges: Let $a_n = \frac{1}{6+\sqrt{n}}$. It is easily verifiable that

$$\lim_{n\to\infty} a_n = 0,$$
$$a_n > 0 \text{ for } n \geq 1$$
$$a_n > a_{n+1} > 0.$$

So, by the Alternating Series Test, the series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

Now, let's show that $\sum_{n=1}^{\infty} a_n$ diverges: Note that for $n \geq 1$,

$$\frac{1}{6+\sqrt{n}} \geq \frac{1}{7\sqrt{n}}.$$

By the p-test for series($p = \frac{1}{2}$), $\sum_{n=1}^{\infty} \frac{1}{7\sqrt{n}}$ diverges. So, by the (Direct) Comparison Test $\sum_{n=1}^{\infty} a_n$ diverges.

6. (continued) Here is a reproduction of the instructions for the problem:
Determine whether each of the following series converge conditionally, converges absolutely, or diverges and circle the appropriate answer. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

b. [7 points]

$$\sum_{n=1}^{\infty} \frac{n^2 + 50n\sin 2n}{n^{7/2}}$$

Circle one: **Absolutely Converges** **Conditionally Converges** **Diverges**

Solution: Set $a_n = \dfrac{n^2 + 50n\sin 2n}{n^{7/2}}$. Note that a_n can be positive or negative, but does not alternate. We have for $n \geq 1$,

$$|a_n| \leq \frac{n^2 + 50n}{n^{7/2}} \leq \frac{51n^2}{n^{7/2}} = 51n^{-3/2}.$$

By the p-test for series ($p = 3/2$, $\sum_{n=1}^{\infty} \frac{51}{n^{3/2}}$ converges. So, by the (Direct) Comparison Test, $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges. So $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

7. [8 points] Suppose F is a nonnegative function defined for all real numbers x. Below are properties of F. Circle **all** that apply to F based on the fact it has this property.

a. [2 points] $\displaystyle\int_{-\infty}^{\infty} F(x)dx = 1$.

(A) F could be a PDF.
(B) F could be a CDF.
(C) F is definitely not a PDF or a CDF.

b. [2 points] $\lim_{x\to\infty} F(x) = 1$ and $F(2) < F(1)$.

(A) F could be a PDF.
(B) F could be a CDF.
(C) F is definitely not a PDF or a CDF.

c. [2 points] $F'(x) > 0$ for $x \geq 0$.

(A) F could be a PDF.
(B) F could be a CDF.
(C) F is definitely not a PDF or a CDF.

d. [2 points] $F(7) = 2$.

(A) F could be a PDF.
(B) F could be a CDF.
(C) F is definitely not a PDF or a CDF.

8. [9 points] The parts of this problem are unrelated.

a. [4 points] Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a geometric series with $a_3 = 54$ and $a_6 = -2$. Write a general formula for a_n:

Solution: Since a_n are the terms of a geometric series, $a_3 = ab^3$ and $a_6 = ab^6$ for some a, b that we want to solve for. To solve for b:

$$\frac{a_6}{a_3} = \frac{-2}{54} = -\frac{1}{27} = b^3$$

So, $b = \frac{-1}{3}$. Now, to solve for a:

$$a_3 = 54 = a\left(\frac{-1}{3}\right)^3 = \frac{-a}{27}.$$

So, $a = -27 * 54$. This gives a general formula for $a_n = -27 * 54\left(\frac{-1}{3}\right)^n$

b. [5 points] Let $b_n = \frac{n}{n+1}$ and $s_n = \displaystyle\sum_{i=1}^{n} b_i$. Circle all statements which are true.

(A) The sequence b_n is bounded.
(B) The sequence b_n is monotone.
(C) $\lim_{n\to\infty} b_n$ exists.
(D) The sequence s_n is bounded.
(E) The sequence s_n is monotone.
(F) $\lim_{n\to\infty} s_n$ exists.

9. [20 points] Otto would like to landscape his yard, so he contacts the company Granville's Calculate-Yourself Gardening. Granville's company provides each potential customer with a list of possible equations and improper integrals guiding the landscaping of the yard. Granville's company also offers a discount to customers that can correctly solve the integrals. Otto, strapped for cash, desperately wants to solve the equations Granville has sent him.

a. [3 points] Granville has also informed Otto that he can plant a maple tree in the back of the yard that is initially 2 meters tall. Granville estimates that the maple tree will grow at an instantaneous rate of

$$M(t) = \frac{12t}{e^t}$$

meters per year t years after it is planted. Write an integral that gives the height of the tree t years after it is planted. Your answer should not involve the letter M.

Solution:

$$2 + \int_0^t \frac{12s}{e^s} ds.$$

b. [7 points] Determine the maximum height that the maple tree will grow to.

Solution: Let us first calculate the integral in (a) using integration by parts ($u = 12s, dv = e^{-s}ds$):

$$\begin{aligned} \int_0^t \frac{12s}{e^s} ds &= -12se^{-s}\big|_0^t + 12\int_0^t e^{-s} ds \\ &= -12te^{-t} + 12\Big[-e^{-s}\big|_0^t\Big] \\ &= -12te^{-t} - 12e^{-t} + 12 \end{aligned}$$

Since the tree is always growing, the maximum height is

$$\begin{aligned} 2 + \int_0^\infty \frac{12t}{e^t} dt &= 2 + \lim_{t\to\infty} \int_0^t \frac{12s}{e^s} ds \\ &= 2 + \lim_{t\to\infty} 12 - 12e^{-t} - 12te^{-t} \\ &= 14 - \lim_{t\to\infty} \frac{12t}{e^t} \\ &\overset{\text{L'H}\frac{\infty}{\infty}}{=} 14 - \lim_{t\to\infty} \frac{12}{e^t} \\ &= 14. \end{aligned}$$

So, the maximum height of the tree is 12 meters.

9. (continued)

c. [10 points] Granville tells Otto that he can get a truck to come and dispense dirt into the yard for 2 hours. The instantaneous rate that the truck will dispense dirt t hours after the truck arrives is

$$D(t) = \frac{(\sin(t))^2}{t^{5/2}\sqrt{2-t}}$$

pounds per minute. Show that $\int_0^2 D(t)dt$ converges. Justify all of your work.

Hint 1: Try splitting this into 2 integrals, one from 0 to 1, and the other from 1 to 2.

Hint 2: You may want to use the fact that $\sin t \leq t$ *for* $t \geq 0$.

Solution: Following the hint,

$$\int_0^2 D(t)dt = \int_0^1 D(t)dt + \int_1^2 D(t)dt.$$

Starting with the first integral, note that on $[0,1]$,

$$D(t) \leq \frac{t^2}{t^{5/2}} \leq t^{-1/2}$$

By the p-test ($p = \frac{1}{2}$), $\int_0^1 \frac{1}{\sqrt{t}}dt$ converges, so by the comparison test, $\int_0^1 D(t)dt$ converges.

Next, we perform a change of variables on the second integral($w = 2 - t$):

$$\begin{aligned}\int_1^2 D(t)dt &= \lim_{b\to 2}\int_1^b D(t)dt \\ &= \lim_{b\to 2} -\int_1^{2-b} \frac{(\sin(2-w))^2}{(2-w)^{5/2}\sqrt{w}}dw \\ &= \int_0^1 \frac{(\sin(2-w))^2}{(2-w)^{5/2}\sqrt{w}}dw\end{aligned}$$

On the interval $[0,1]$,

$$\frac{(\sin(2-w))^2}{(2-w)^{5/2}\sqrt{w}} \leq \frac{1}{\sqrt{w}}$$

By the p-test ($p = \frac{1}{2}$), $\int_0^1 \frac{1}{\sqrt{w}}dw$ converges. So, by the comparison test, $\int_1^2 D(t)dt$ converges. As both parts converge, we have verified that the integral $\int_0^2 D(t)dt$ converges.