Math 116 — Final Exam — December 14, 2021

Write your 8-digit UMID number very clearly in the box to the right.

Your Initials Only: ______ Instructor Name: ______________________ Section #: ____

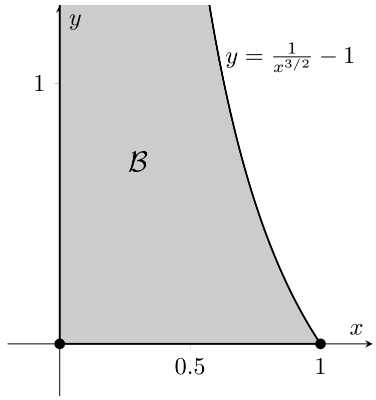
1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 13 pages including this cover.
3. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	9	
2	5	
3	7	
4	6	
5	9	
6	13	

Problem	Points	Score
7	17	
8	16	
9	7	
10	11	
Total	100	

1. [9 points] Arnold is building a set for his son Michael's school play in which Michael will have to climb a very tall beanstalk to fight a giant.

a. [4 points] At first, Arnold decides that since the beanstalk is extremely tall, he should model it as an infinitely tall solid of revolution of the region $\mathcal{B}$ around the y-axis. Here, $\mathcal{B}$ is the unbounded region in the first quadrant to the left of the function $f(x) = \frac{1}{x^{3/2}} - 1$ for $0 < x \leq 1$, depicted partially below.



Write an integral for just the **area** of the region $\mathcal{B}$ (and not the rotated solid) in the space below. Determine whether your integral converges or diverges, with FULL JUSTIFICATION, and circle the word CONVERGES or DIVERGES corresponding to your conclusion.

Solution: The integral is

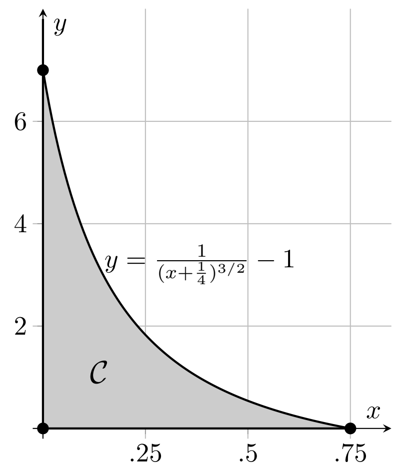
$$\int_0^1 \frac{1}{x^{3/2}} - 1dx = \int_0^1 \frac{1}{x^{3/2}}dx - 1.$$

Using the p-test ($p = \frac{3}{2}$), the integral $\int_0^1 \frac{1}{x^{3/2}}dx$ diverges, so the whole integral diverges.

The integral is $\int_0^1 \frac{1}{x^{3/2}} - 1dx$ and it CONVERGES / $\boxed{\text{DIVERGES}}$.

1. (continued)

b. [5 points] Arnold realizes modelling a beanstalk as infinitely tall is not the most realistic, so he changes his region to be $\mathcal{C}$. Here, the region $\mathcal{C}$ is bounded by the function $g(x) = \frac{1}{(x+\frac{1}{4})^{3/2}} - 1$, the x-axis, and the y-axis, depicted below.



If the model of the Beanstalk is now the solid formed by rotating the the region $\mathcal{C}$ around the y-axis, write, but do not solve, an integral that gives the **volume** of the beanstalk using the blank provided.

Solution: Taking vertical slices, we see that we obtain the shell method. The volume of a slice of thickness Δx at a horizontal coordinate x is approximately

$$\Delta V = 2\pi x \left(\frac{1}{(x+\frac{1}{4})^{3/2}} - 1 \right) \Delta x,$$

and so the total volume of the solid is

$$\int_0^{.75} 2\pi x \left(\frac{1}{(x+\frac{1}{4})^{3/2}} - 1 \right) dx.$$

Alternate Solution: Taking horizontal slices, we see that we obtain the disc method. The volume of a slice of thickness Δy at vertical coordinate y is approximately

$$\Delta V = \pi x^2 \Delta y = \pi \left((y+1)^{-2/3} - \frac{1}{4} \right)^2 \Delta y$$

and so now the total volume of the solid is

$$\int_0^{7} \pi \left((y+1)^{-2/3} - \frac{1}{4} \right)^2 dy.$$

The integral is $\underline{\int_0^{.75} 2\pi x \left(\frac{1}{(x+\frac{1}{4})^{3/2}} - 1 \right) dx}$.

2. [5 points] Suppose that the power series $\sum_{n=1}^{\infty} C_n(x+2)^n$ converges at $x=4$ and diverges at $x=-10$. What can you say about the behavior of the power series at the following values of x?

a. [1 point] At $x=0$, the power series...

$\boxed{\text{CONVERGES}}$ DIVERGES CANNOT DETERMINE

b. [1 point] At $x=-8$, the power series...

CONVERGES DIVERGES $\boxed{\text{CANNOT DETERMINE}}$

c. [1 point] At $x=8$, the power series...

CONVERGES $\boxed{\text{DIVERGES}}$ CANNOT DETERMINE

d. [1 point] At $x=-4$, the power series...

$\boxed{\text{CONVERGES}}$ DIVERGES CANNOT DETERMINE

e. [1 point] At $x=6$, the power series...

CONVERGES DIVERGES $\boxed{\text{CANNOT DETERMINE}}$

3. [7 points] A function $F(x)$ has Taylor series given by

$$F(x)=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{2^n(n^2+1)}(x-1)^{4n+1}$$

Answer the following questions regarding the Taylor series:

a. [3 points] Circle the appropriate answer. No work is needed, but partial credit may be given for correct work.

At $x=1$, F is... $\boxed{\text{INCREASING}}$ DECREASING CANNOT DETERMINE

Solution: Using the Taylor series,

$$F'(1)=\frac{(-1)^0(1)}{2^0(1)}=1>0,$$

so F is increasing at $x=1$.

b. [4 points] What is $F^{(2021)}(1)$? Give your answer in exact form and do not try to simplify. Show your work.

Solution: Using the Taylor series, the term $(x-1)^{2021}$ appears when $n=505$, so

$$\frac{F^{(2021)}(1)}{2021!}=\frac{(-1)^{505}(505+1)}{2^{505}(505^2+1)}$$

$F^{(2021)}(1) =$ $\underline{-\frac{(506)(2021)!}{2^{505}(505^2+1)}}$.

4. [6 points] Find the **radius of convergence** of the following power series:

$$\sum_{n=0}^{\infty} \frac{8^n (n!)^3}{(3n)!}(x-5)^{3n}.$$

Show your work including full justifications of any tests you use.

Solution: Setting $a_n = \frac{8^n (n!)^3}{(3n)!}(x-5)^{3n}$, compute

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{8^{n+1}\left((n+1)!\right)^3}{(3n+3)!}\frac{(3n)!}{8^n (n!)^3}|x-5|^3 \\
&= \lim_{n\to\infty} 8\frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}|x-5|^3 \\
&= \frac{8}{27}|x-5|^3.
\end{aligned}$$

By the ratio test, the power series converges for

$$\frac{8}{27}|x-5|^3 < 1 \iff |x-5| < \left(\frac{27}{8}\right)^{1/3} = \frac{3}{2}.$$

The radius of convergence is $\frac{3}{2}$.

5. [9 points] The radius of convergence of the power series below is 4. Find the **interval of convergence**.

$$\sum_{n=1}^{\infty} \frac{5n}{4^n(n^2+1)}(x+1)^n.$$

Show all your work including full justifications of convergence and divergence of any relevant series. Throughout this problem, you may assume that the radius of convergence is 4 and you do not need to recompute it.

Solution: Observing the power series, the center of the interval of convergence is $x = -1$. Using the radius of convergence, the endpoints are $x = -5$ and $x = 3$.

At $x = -5$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n 5n}{(n^2+1)}.$$

Setting $a_n = \frac{5n}{n^2+1}$, $a_n > 0$, $\lim_{n\to\infty} a_n = 0$, and $a_{n+1} < a_n$. So, by the Alternating Series Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 5n}{(n^2+1)}$ converges.

At $x = 3$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{5n}{(n^2+1)}.$$

For $n \geq 1$,

$$\frac{5n}{n^2+1} \geq \frac{1}{n},$$

and by the p-test ($p = 1$), $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So, by the Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{5n}{(n^2+1)}$ diverges.

The interval of convergence is $[-5, 3)$.

6. [13 points] Values of a function $g(x)$ and some of its derivatives at $x = 2$ are given in the table below. Use this information for some of the problems below.

$g(2)$	$g'(2)$	$g''(2)$	$g'''(2)$	$g^{(4)}(2)$
1	2	-4	0	4

a. [4 points] Find the first 4 nonzero terms of the Taylor series of $g(x)$ about $x = 2$. Write your final answer as a polynomial $P(x)$ in the blank below.

$P(x) =$ $1 + 2(x-2) - 2(x-2)^2 + \frac{1}{6}(x-2)^4$

b. [4 points] Using known Taylor series, find the first 3 nonzero terms of the Taylor series of $f(x) = (x-2)\ln\left(\frac{x}{2}\right)$ about $x = 2$. Write your final answer as a polynomial $Q(x)$ in the blank below. (*Hint:* $f(x) = (x-2)\ln\left(1 + \frac{(x-2)}{2}\right)$)

Solution: For $-1 < x \leq 1$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots,$$

about $x = 0$. So,

$$(x-2)\ln\left(1 + \frac{(x-2)}{2}\right) = (x-2)\left(\frac{x-2}{2} - \frac{1}{2}\left(\frac{x-2}{2}\right)^2 + \frac{1}{3}\left(\frac{x-2}{2}\right)^3 + \dots\right)$$

$Q(x) =$ $\frac{(x-2)^2}{2} - \frac{(x-2)^3}{8} + \frac{(x-2)^4}{24}$

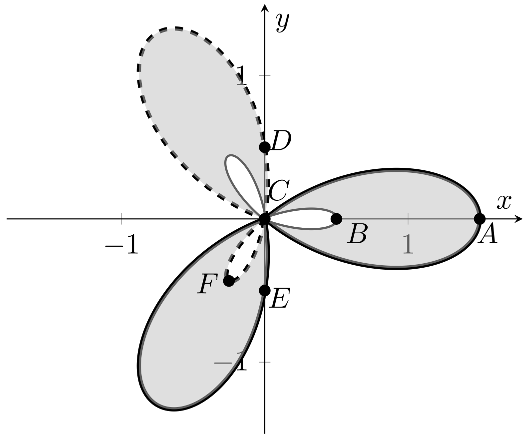
c. [5 points] Let $H(x) = 1 + \int_2^x f(t) + g(t)dt$. Find the first 4 nonzero terms of the Taylor series of H about $x = 2$. Write your final answer as a polynomial $R(x)$ in the blank below. Partial credit may be given for finding the appropriate terms of $\int_2^x f(t)dt$ or $\int_2^x g(t)dt$.

Solution: Putting together (a) and (b), R is the first 4 nonzero terms of $1 + \int_2^x P(t) + Q(t)dt$. Note that we only need the first 3 terms of P and the first term of Q:

$$\begin{aligned} 1 + \int_2^x P(t) + Q(t)dt &= 1 + \left[t + (t-2)^2 - \frac{2}{3}(t-2)^3\right]\Big|_2^x + \left[\frac{(t-2)^3}{6}\right]\Big|_2^x \\ &= 1 + (x-2) + (x-2)^2 - \frac{2}{3}(x-2)^3 + \frac{(x-2)^3}{6}. \end{aligned}$$

$R(x) =$ $1 + (x-2) + (x-2)^2 - \frac{(x-2)^3}{2}$

7. [17 points] John is holding a Fan Fair to celebrate the success of his burgeoning fan business. At the fair, John is debuting his new fan, which has blades given by the shaded region of the graph of the polar equation $r = \cos(3\theta) + \frac{1}{2}$ shown below. Note that the graph of $r = \cos(3\theta) + \frac{1}{2}$ is comprised of both the inner and outer loops of the fan blades. One of the activities at the Fan Fair is to guess the perimeter and area of the blades, which can actually be computed explicitly.



a. [4 points] For the values of θ listed below, write on the line the letter of the point corresponding to it.

$\theta = 0$: A $\theta = \frac{\pi}{3}$: F

$\theta = \frac{\pi}{2}$: D $\theta = \pi$: B

b. [5 points] Find the 3 values of θ which correspond to the point C (the origin) for $0 \le \theta \le \pi$. Then, determine the interval within $[0, 2\pi]$ for which θ traces out the **dashed** loops in the graph above. (*Hint:* $\cos\frac{2\pi}{3} = \cos\frac{4\pi}{3} = -\frac{1}{2}$)

Solution: Note that the solutions to $\cos(3\theta) = -\frac{1}{2}$ are, for k an integer

$$3\theta = \frac{2\pi}{3} + 2\pi k$$
$$3\theta = \frac{4\pi}{3} + 2\pi k.$$

So, the first three values are

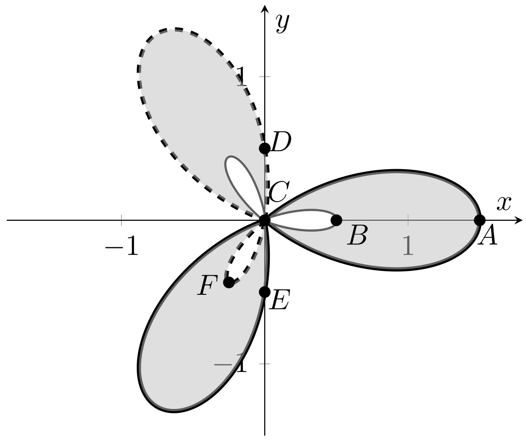
$$3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}.$$

Using (a), we know that the dashed loops lie between the first value and the third value when θ corresponds to the point C.

$\theta =$ $\frac{2\pi}{9}$, $\frac{4\pi}{9}$, $\frac{8\pi}{9}$

Interval giving θ-values that trace out the dashed loops: $[\frac{2\pi}{9}, \frac{8\pi}{9}]$

7. (continued) Here is a reproduction of the graph from the previous page of the polar equation $r = \cos(3\theta) + \frac{1}{2}$:



c. [4 points] Write, but do not evaluate, an expression involving one or more integrals that gives the total perimeter of the fan blades, including both the inner and outer edges of the fan blades.

Solution: Note that $\frac{dr}{d\theta} = -3\sin(3\theta)$ and the perimeter is given by

$$\int_0^{2\pi} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Total Perimeter = $\int_0^{2\pi} \sqrt{(\cos(3\theta) + \frac{1}{2})^2 + 9\sin^2(3\theta)}\, d\theta$

d. [4 points] Write, but do not evaluate, an expression giving the total area of all 3 fan blades (the shaded region of the graph). (*Hint*: Your answer from (b) may be handy, but is not strictly necessary)

Solution: Using (b), the small dashed loop is traced out for $\frac{2\pi}{9} \le \theta \le \frac{4\pi}{9}$ and the large dashed loop is traced out for $\frac{4\pi}{9} \le \theta \le \frac{8\pi}{9}$, so the area of one small loop is

$$\int_{\frac{2\pi}{9}}^{\frac{4\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2}\, d\theta$$

and the area of one large loop is

$$\int_{\frac{4\pi}{9}}^{\frac{8\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2}\, d\theta.$$

Exploiting the symmetry of the fan, we get the total area below.

Total Area = $3\left[\int_{\frac{4\pi}{9}}^{\frac{8\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2}\, d\theta - \int_{\frac{2\pi}{9}}^{\frac{4\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2}\, d\theta\right]$

8. [16 points] Molly and Erin are two bumper car enthusiasts who hate bumping into things. So, they get on the bumper cars, ride until they bump into each other, and then stop riding the bumper cars. At time t minutes after they start driving their bumper cars, Molly's position is given by

$$M(t) = \left(4 - 6\cos t, 3t + \pi^2\right)$$

and Erin's position is given by

$$E(t) = \left(2\cos t, (t-2\pi)^2 + 3t + \frac{8\pi^2}{9}\right),$$

where all distances are in meters.

a. [4 points] How long do they ride the bumper cars? Make sure to include units.

Solution: We set

$$4 - 6\cos t = 2\cos t \quad \text{and} \quad 3t + \pi^2 = (t-2\pi)^2 + 3t + \frac{8\pi^2}{9},$$

we get

$$\cos t = \frac{1}{2} \quad \text{and} \quad (t-2\pi)^2 = \frac{\pi^2}{9},$$

so the x-coordinates are equal when $t = \frac{\pi}{3}, \frac{5\pi}{3} + 2\pi k$ and the y-coordinates are equal when $t = \frac{5\pi}{3}, \frac{7\pi}{3}$. So, they ride the bumper cars for $\frac{5\pi}{3}$ minutes.

b. [4 points] Find an explicit expression for Erin's speed t minutes after she starts driving her bumper cars, before the collision (your expression should not contain any integrals nor the letters M, E). Make sure to include units.

Solution: For Erin, $\frac{dx}{dt} = -2\sin t$ and $\frac{dy}{dt} = 2(t-2\pi)+3$. So, her speed is

$$\sqrt{4\sin^2 t + (2(t-2\pi)+3)^2} \text{ meters/minute}$$

c. [4 points] Write, but do not evaluate, an expression involving integrals that gives the total distance that Erin travelled before the collision. Make sure to include units.

Solution: Using the answer to (a),

$$\int_0^{\frac{5\pi}{3}} \sqrt{4\sin^2 t + (2(t-2\pi)+3)^2 dt} \text{ meters}$$

d. [4 points] Suppose the positive y-direction in the xy-plane is North. At $t = 0$, Molly is facing directly North. Find all other times $t > 0$ (if any) after they start riding their bumper cars, but before their collision, when Molly is facing directly North. Make sure to include units.

Solution: For, Molly $\frac{dx}{dt} = 6\sin t$ and $\frac{dy}{dt} = 3$. So, $\frac{dx}{dt} = 0$ when $t = k\pi$. The only $t > 0$ before the collision where $\frac{dx}{dt} = 0$ is π minutes.

9. [7 points] Emily is transporting a chocolate ice cream cone up 1 story of East Hall to her friend. However, there is a hole in the bottom of the cone and ice cream drips out in a steady stream.

The mass of the cone is 200 grams and there are initially 100 grams of ice cream in the cone. The ice cream drips out at a rate of 4 grams/sec. Emily spends 10 seconds raising the cone at a constant rate of 0.5 m/sec to reach her friend.

a. [3 points] What is the total mass in grams of the cone and the ice cream in the cone when Emily has lifted it a vertical distance ℓ m?

Solution: When Emily has lifted the cone a vertical distance of ℓ m, she has lifted for 2ℓ sec. Therefore, 8ℓ grams of ice cream has dripped out of the cone. The cone and ice cream start at 300 grams.

Mass = $300 - 8\ell$ grams

b. [4 points] Write, but do not evaluate, an integral that represents the total amount of work (in grams m^2/sec^2) done by Emily lifting the cone filled while the ice cream drips. You may assume the acceleration due to gravity is $g = 9.8$ m/sec^2.

Solution: The work to lift the rocket between ℓ and $\ell + \Delta\ell$ m is

$$\Delta W = 9.8(300 - 8\ell)\Delta\ell.$$

Hence, the total work is below.

Work = $\int_0^5 9.8(300 - 8\ell)\,d\ell$ grams m^2/sec^2

10. [11 points] Every year, Tommy cooks a feast for his friends and asks them to rate the feast on a scale from 0 to 5. The total average of the ratings from all of his feasts is his *Cumulative Feast Rating.*

After the first feast, Tommy's Cumulative Feast Rating is 4. In the nth year of the feasts, his Cumulative Feast Rating changes by $a_n = 2(-1)^n\left(\frac{1}{2}\right)^n$ from the previous year. That is, if R_n is the Cumulative Feast Rating after the nth feast, $R_1 = 4$ and $R_n = R_{n-1} + a_n$ for $n = 2, 3, 4, \ldots$.

a. [3 points] Tommy wants his feasts to be a success! What will his Cumulative Feast Rating be if he hosts feasts forever? Be sure to show your work.

Solution: The Cumulative Feast Rating tends to

$$4 + \sum_{n=2}^{\infty} 2\left(\frac{-1}{2}\right)^n = 4 + 2\left(\frac{1}{1+\frac{1}{2}} - 1 + \frac{1}{2}\right) = 4 + \frac{1}{3}.$$

b. [3 points] Tommy, knowing he will be unable to supply his friends with feasts forever, so he is interested in how many feasts he will have to host to be within .05 of his final Cumulative Feast Rating. Circle the values of n for which R_n will be within .05 of the value from part (a).

(A) $n = 1$

(B) $n = 3$

$\boxed{\text{(C)}}$ $n = 4$

$\boxed{\text{(D)}}$ $n = 25$

c. [5 points] Suppose that P is the probability density function that Tommy's friends finish the cookies he has prepared for the feast t minutes after he sets them out. Suppose P is given by the following for some $a > 0, b > 0$:

$$P(t) = \begin{cases} bt & 0 < t < a \\ 0 & \text{otherwise} \end{cases}$$

If the median time the cookies take to be finished is 20 minutes, find a and b so that P is a pdf.

Solution: Since P is a pdf, $\int_0^a bt\,dt = 1$. Since the median is 20 minutes, $\int_0^{20} bt\,dt = \frac{1}{2}$, giving

$$\frac{1}{2} = \left.\frac{bt^2}{2}\right|_0^{20} = 200b \Rightarrow b = \frac{1}{400}.$$

Plugging into the first equation, we get

$$1 = \left.\frac{1}{800}t^2\right|_0^a = \frac{a^2}{800} \Rightarrow a = \sqrt{800}.$$

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin\theta$	$\cos\theta$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$