Math 116 — Second Midterm — November 15, 2022

Write your 8-digit UMID number very clearly in the box to the right.

Your Initials Only: ______ Instructor Name: ____________________ Section #: ____

1. This exam has 10 pages including this cover.
2. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. You are allowed notes written on two sides of a $3'' \times 5''$ note card.
6. You are NOT allowed other resources, including, but not limited to, notes, calculators or other devices.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	8	
3	15	
4	9	
5	12	
6	11	

Problem	Points	Score
7	7	
8	5	
9	8	
10	14	
Total	100	

1. [11 points]

a. [7 points] Determine the **radius** of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{9^n (x-2)^{2n}}{n^2}$$

Be sure to show all of your work. Write your final answer in the space provided below.

Answer: ____________________.

b. [4 points] Suppose that the power series

$$\sum_{n=1}^{\infty} a_n (x-5)^n$$

converges when $x = 10$ and diverges when $x = -1$. At which of the following x-values must the series converge? Circle your answers. You do not need to show any work for this problem.

-5 $\quad$ 0 $\quad$ 2 $\quad$ 5 $\quad$ 11 $\quad$ 12

2. [8 points] Suppose that a_n, b_n, and c_n are sequences with the following properties:

- The sequence a_n is bounded
- The series $\sum_{n=1}^{\infty} b_n$ converges absolutely
- $\frac{1}{n^2+1} \leq c_n \leq \frac{1}{n}$ for all $n \geq 1$

Determine whether the following statements are **always**, **sometimes**, or **never** true, and circle the appropriate answer for each part. No justification is necessary.

a. [2 points] The sequence b_n converges to 0.

Circle one: **Always** **Sometimes** **Never**

b. [2 points] $\sum_{n=1}^{\infty} \frac{c_n}{n}$ diverges.

Circle one: **Always** **Sometimes** **Never**

c. [2 points] The sequence a_n converges.

Circle one: **Always** **Sometimes** **Never**

d. [2 points] The series $\sum_{n=1}^{\infty} \frac{1}{n^3 c_n}$ converges.

Circle one: **Always** **Sometimes** **Never**

3. [15 points] Carlos and Nancy are catching a train that leaves at 4pm. They leave their apartment for the train station at 12pm. The amount of time t (in hours) that elapses between the time they leave their apartment and the time they arrive at the train station is described by the following **probability density function** (pdf) $h(t)$:

$$h(t) = \begin{cases} 0 & t \leq 3 \\ a(t-3) & 3 < t \leq 4 \\ \frac{1}{4}e^{4-t} & 4 < t < \infty. \end{cases}$$

a. [5 points] What is the probability they arrive late for their train (i.e., what is the probability they arrive at the train station after 4pm)? Be sure to show work for your calculations, and be sure to use proper notation.

b. [4 points] Find the value of a so that $h(t)$ is a probability density function. Be sure to show work for any calculations.

c. [3 points] Give a practical interpretation of the fact that $h(4.5) = 0.15$. Note that the output value has been rounded to the nearest hundredth.

d. [3 points] Write an expression involving one or more integrals that gives the mean amount of time it takes Nancy and Carlos to travel to the train station. The letter h should not appear in your answer. You do not need to evaluate any integrals for this part.

4. [9 points] Consider the following sequences, all defined for $n = 1, 2, 3, \ldots$

$$a_n = \int_1^n \frac{1}{x^2}\, dx$$
$$b_n = 1 + \frac{(-1)^n(n+1)}{n^2}$$
$$c_n = \sum_{k=1}^{n} \frac{1}{(k+1)^{0.5}}$$

a. [3 points] Which sequences are monotone? No justification is required for this part of the problem. Circle your final answer(s) below.

Circle your answers: a_n b_n c_n NONE

b. [3 points] Which sequences are convergent? No justification is required for this part of the problem. Circle your final answer(s) below.

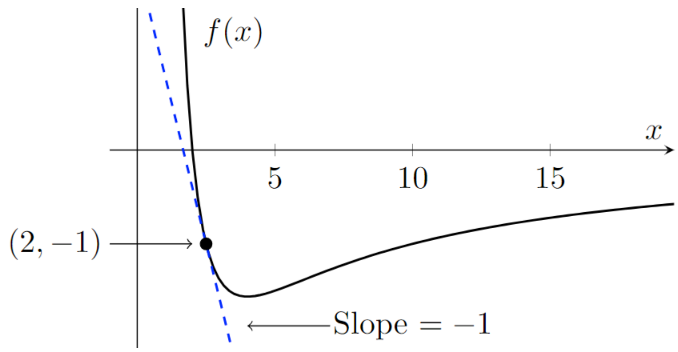
Circle your answers: a_n b_n c_n NONE

c. [3 points] Does the series $\sum_{n=1}^{\infty} a_n$ converge? Justify your answer.

5. [12 points] The solid curve graphed below is part of the graph of a function $f(x)$ which has the following properties:

- $f(x)$ is twice differentiable on the interval $(0, \infty)$.
- $f(2) = -1$.
- For all $x \geq 10$, $f(x) < -\dfrac{5}{x}$.

The dashed line is the tangent line to $f(x)$ at $x = 2$, and its slope is -1.



a. [3 points] Compute $\displaystyle\lim_{x\to 2}\left(\frac{f(x)+1}{\cos\left(\frac{\pi}{2}x\right)+\frac{1}{2}x}\right)$

b. [3 points] Compute $\displaystyle\lim_{x\to\infty} x[f(2+x^{-1})+1]$

c. [6 points] Does the following improper integral converge or diverge? Fully justify your answer including using proper notation and showing mechanics of any tests you use.

$$\int_1^{\infty} (-f(x))\, dx.$$

6. [11 points] It's fall, and the leaves are falling onto the grass. Over the course of each day, suppose that 10kg of leaves fall onto the grass. At the beginning of each day, 50 percent of the leaves currently on the grass (from previous days) are removed.

a. [3 points] Let M_n be the total mass in kg of all the leaves currently on the grass at the **end** of the nth day. Suppose that $M_1 = 10$. Write expressions for M_2 and M_3. The letter M should not appear in your answer.

b. [5 points] Find a closed-form expression for M_n. This means that your answer should be a function of n, should not contain Σ, and should not be recursive.

c. [3 points] If 30kg of leaves are on the grass at the end of any day, then the grass will die. Will this happen during the fall? Justify your answer.

7. [7 points] Determine whether the following improper integral converges or diverges. Circle your final answer choice. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

$$\int_1^\infty \frac{t^2 - \ln(t)}{t^4 + 8t + 10}\, dt.$$

Circle one: **Converges** **Diverges**

8. [5 points] Fully evaluate the following integral:

$$\int x \ln x \, dx$$

You do not need to simplify your answer.

9. [8 points] For each of the following, circle ALL that apply. There may be more than one correct answer for each part. You do not need to show any work for any part of this question.

a. [4 points] Suppose $P(t)$ is a **cumulative distribution function** (cdf) satisfying $P(0.2) = 0.5$. Which of the following MUST be true?

$\lim_{t\to\infty} P(t) = 1$ The median t-value is 0.2.

$P(1) \geq 0.5$ The mean t-value is 0.2.

NONE OF THESE

b. [4 points] The series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n + \sin(n)}{n^{3.1}}$...

DIVERGES CONVERGES CONDITIONALLY

CONVERGES CONVERGES ABSOLUTELY

NONE OF THESE

10. [14 points] Determine if the following series converge or diverge. Circle your final answer choice for each. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

a. [7 points] $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+4n-1}$

Circle one: **Converges** **Diverges**

b. [7 points] $\displaystyle\sum_{n=1}^{\infty}\frac{2n-1}{n^2+n+2}$

Circle one: **Converges** **Diverges**