Math 116 — First Midterm — October 4, 2022

Write your 8-digit UMID number very clearly in the box to the right.

Your Initials Only: ______ Instructor Name: ______________________ Section #: ____

1. This exam has 14 pages including this cover.
2. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. You are allowed notes written on two sides of a $3'' \times 5''$ note card.
6. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
7. Include units in your answer where that is appropriate.
8. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	16	
2	15	
3	12	
4	5	
5	7	
6	10	

Problem	Points	Score
7	13	
8	12	
9	10	
Total	100	

1. [16 points] Use the table to compute the following quantities. The function $h(x)$ is **odd**, **twice differentiable**, and $h'(x) > 0$ for all x-values. Write your answer using exact form on the blank provided. If there is not enough information available to answer the question, write N.I. You need to evaluate all integrals, but you do not need to simplify your final answer.

x	0	1	2	3	4
$h(x)$	0	2	4	5	7
$h'(x)$	7	3	4	7	2

a. [4 points] $\displaystyle\int_3^4 \frac{h''(t)}{h'(t)}\,dt$

Answer: $\ln(2/7)$.

Solution: Let $u = h'(t)$, so $du = h''(t)dt$ and the integral becomes $\displaystyle\int_7^2 u^{-1}\,du = \ln(2) - \ln(7)$ which is $\ln(2/7)$.

b. [4 points] The average value of $h'(x)$ on $[-1, 1]$

Answer: **2**.

Solution: The average value of $h'(x)$ on $[-1, 1]$ is by definition $\displaystyle\frac{1}{1-(-1)}\int_{-1}^{1} h'(x)\,dx$. Using the fundamental theorem of calculus, this is equal to $\frac{1}{2}(h(1) - h(-1))$. Since $h(x)$ is odd, $h(-1) = -h(1) = -2$, so the answer is $\frac{1}{2}(2 - (-2)) = 2$.

c. [4 points] $\displaystyle\int_1^4 (w + 1)h''(w)\,dw$

Answer: -1.

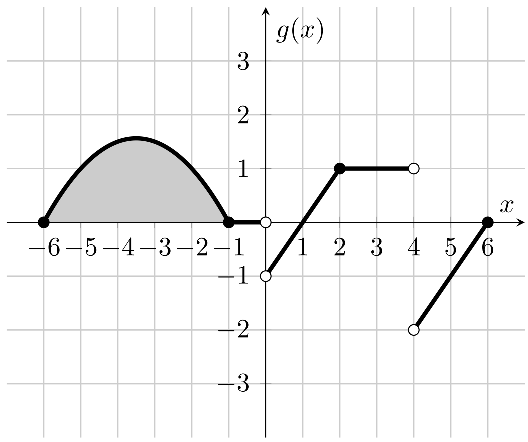
Solution: Since integration splits over addition, this is equal to $\displaystyle\int_1^4 wh''(w)\,dw + \int_1^4 h''(w)\,dw$. For the first integral we integrate by parts to obtain $\displaystyle\int_1^4 wh''(w)\,dw = -\int_1^4 h'(w)\,dw + wh'(w)|_1^4 = h(1) - h(4) + 4h'(4) - h'(1) = 0$. For the second integral we use FTC to get $\displaystyle\int_1^4 h''(w)\,dw = h'(4) - h'(1) = 2 - 3 = -1$.

d. [4 points] $\displaystyle\int_{1/2}^{2} x^{-1/2}h'(\sqrt{2x})\,dx$

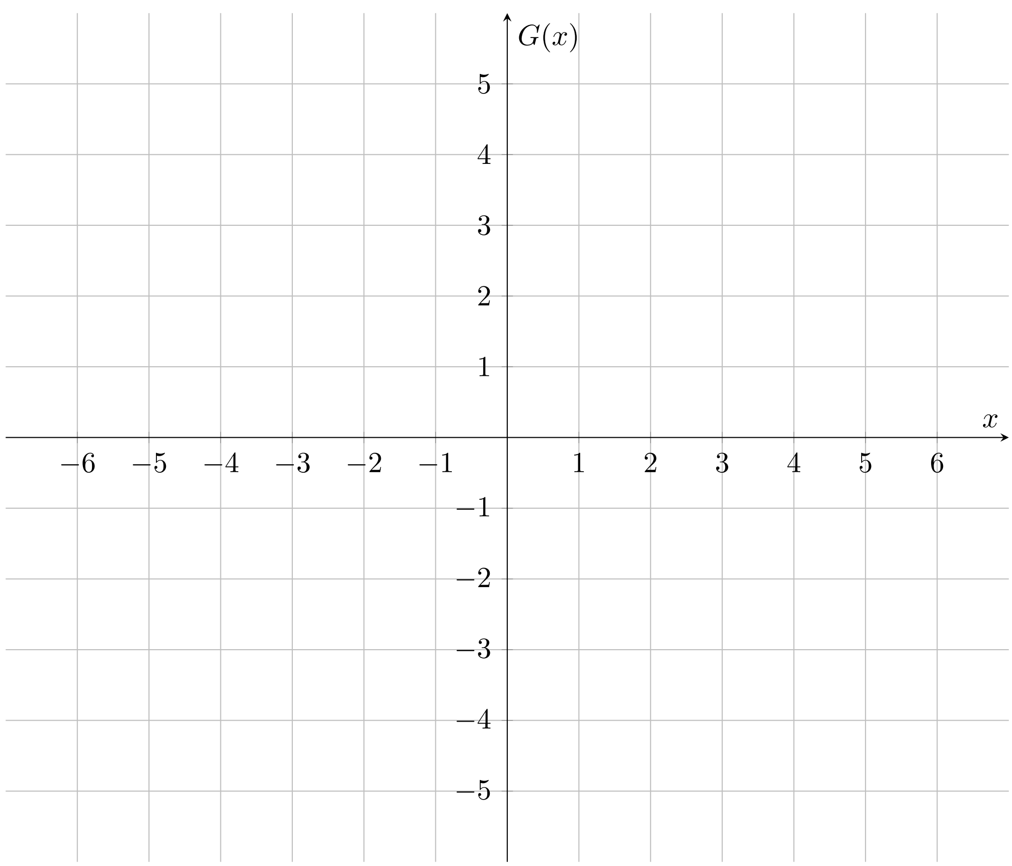
Answer: $2\sqrt{2}$.

Solution: Let $u = \sqrt{2x} = (2x)^{1/2}$. By the chain rule and power rule, $du = \frac{1}{\sqrt{2x}}dx$. Therefore $dx = \sqrt{2x}du$. So the integral given is equal to $\sqrt{2}\int_1^2 h'(u)\,du = \sqrt{2}(h(2) - h(1)) = 2\sqrt{2}$.

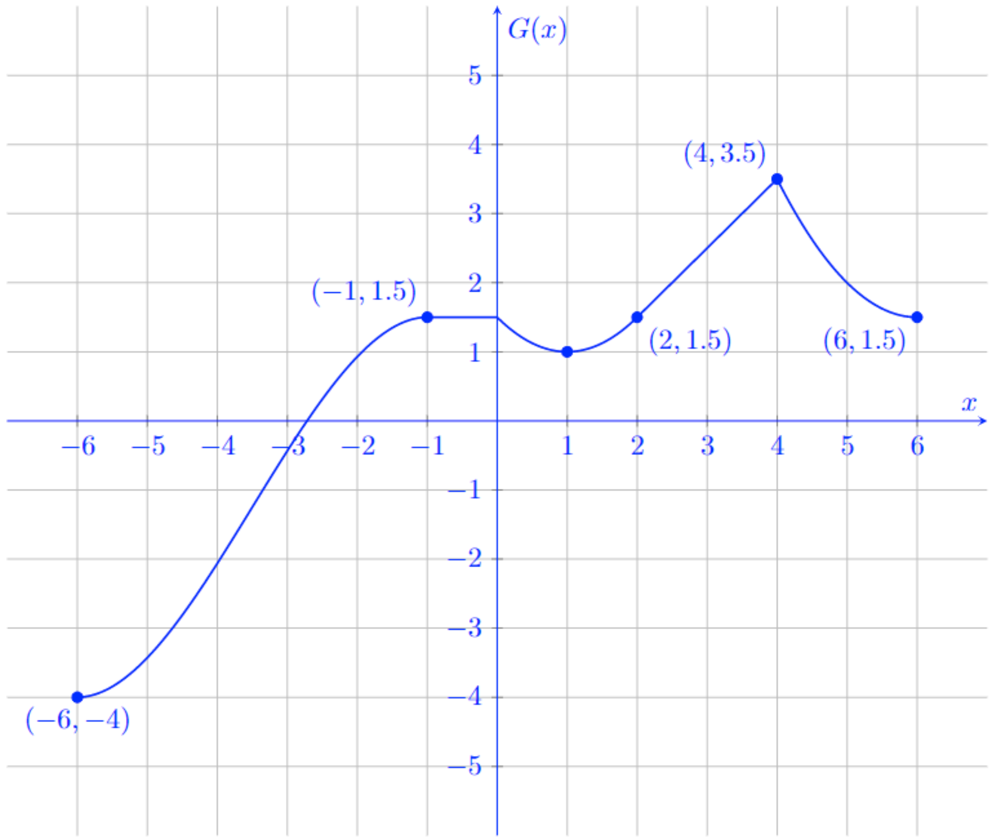
2. [15 points] The function $g(x)$ is graphed below. The area of the shaded region is 5.5. The function $g(x)$ is piecewise linear for $x > -1$.



On the axes provided below, sketch a continuous antiderivative $G(x)$ of $g(x)$ with domain $[-6, 6]$, satisfying $G(1) = 1$. Make sure to clearly label the input and output values at $x = -6, -1, 2, 4$, and 6. Be sure to make it clear where $G(x)$ is **concave up**, **concave down**, or **linear**, and where it is **increasing**, **decreasing**, or not **differentiable**.



Solution: The input/output values at the specified points are labeled in the figure. The graph of $G(x)$ should be concave up on $(-6,-3.5)$, $(0,2)$, and $(4,6)$, concave down on $(-3.5,-1)$, and linear on $(-1,0)$ and $(2,4)$. The function $G(x)$ is increasing on $(-6,-1)$ and $(1,4)$ and decreasing on $(0,1)$ and $(4,6)$. The function $G(x)$ is not differentiable at $(0,1.5)$ and $(4,3.5)$.



3. [12 points] In each part of this problem, circle all options which make the statement **always true**. There may be more than one correct answer for each part. If none of the options make the statement always true, then circle NONE OF THESE. You must circle your answers entirely to receive credit on this problem.

a. [3 points] If $f(x)$ and $g(x)$ are differentiable functions satisfying $f(0) = 0$ and $f'(x) = g(x)$, then

$g(0) = 0.$

$\boxed{\pi + f(x) \text{ is an antiderivative of } g(x).}$

$\boxed{\int_0^1 g(x)dx = f(1).}$

$2^e + g(x)$ is an antiderivative of $f(x)$.

NONE OF THESE

b. [3 points] Suppose $a < b$, and that $p(x)$ and $r(x)$ are continuous functions on $[a, b]$. If $\int_a^b p(x)\,dx < \int_a^b r(x)\,dx$ then

$p(x) < r(x)$ for all x in the interval $[a, b]$.

$p(a) < r(a)$.

$p(x) > r(x)$ for at least one value x in $[a, b]$.

$p(0) = 0$.

$\boxed{\text{NONE OF THESE}}$

c. [3 points] Suppose $k(t)$ is a continuous function with $k(t) < 0$ for all t, If $M(w) = \int_0^{-w^2} k(t)\,dt$ then

$\boxed{M(w) \text{ is increasing for } w > 0.}$

$\boxed{M(0) = 0.}$

$M(w)$ is increasing for $w < 0$.

$\boxed{M(w) \text{ is decreasing for } w < 0.}$

NONE OF THESE

d. [3 points] If $h(x) = x^2 + x + 1$, consider the integral $I = \int_0^{10} h(x)\,dx$. The integral I is

$\boxed{\text{larger than MID(5).}}$

larger than TRAP(5).

$\boxed{\text{larger than LEFT(5).}}$

larger than RIGHT(5).

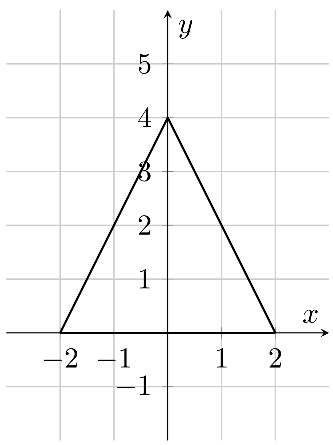
NONE OF THESE

4. [5 points] Find the derivative of $f(x) = 3e^{-2x}\cos(5x)$. You do not need to simplify your answer.

Solution: We use the chain rule and the product rule:

$$\begin{aligned} f'(x) &= 3\frac{d}{dx}(e^{-2x}\cos(5x)) \\ &= 3\Big[-2e^{-2x}\cos(5x) - 5e^{-2x}\sin(5x)\Big] \\ &= -6e^{-2x}\cos(5x) - 15e^{-2x}\sin(5x). \end{aligned}$$

5. [7 points] Nzinga is going rock climbing at a local climbing gym. The gym building is shaped as follows. Its base is the triangular region shown in the figure below. The cross sections of the gym perpendicular to the y-axis are semicircles.



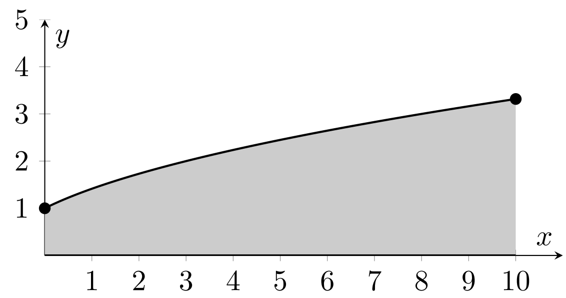
Write, but do not evaluate, an integral which gives the volume enclosed by the building.

Solution: The equation describing the right slanted line segment in the diagram is $y = -2x+4$, and the equation describing the left slanted line segment is $y = 2x+4$. The base of a semicircular slice which is at a height y above the x-axis is given by the difference in x-coordinate of the two slanted segments at the height y. This is $\frac{4-y}{2} - \frac{y-4}{2} = 4-y$.
The radius of the semicircular slice is then $(4-y)/2$, so the area is $\frac{1}{2}\pi((4-y)/2)^2 = \frac{\pi(4-y)^2}{8}$.
So the volume of the slice, with thickness Δy, is $\frac{\pi(4-y)^2}{8}\Delta y$. Therefore the volume is

$$\int_0^4 \frac{\pi(4-y)^2}{8}\,dy.$$

6. [10 points] Denise and Trystan are undersea research scientists, and they are preparing to descend into the ocean in a newly-constructed submarine. The submarine's shape is given by rotating the region below the curve $y = \sqrt{x+1}$, above the x-axis, and between $x = 0$ and $x = 10$ (see figure) about the x-axis. Here, x and y are measured in meters.

Graph of $y = \sqrt{x+1}$ from $x = 0$ to $x = 10$



The density of the submarine is not constant, due to the advanced materials used in its construction. Instead, the density $p(x)$ varies, and is given by $p(x) = (x-5)^2 + 1$ kg/m^3.

a. [5 points] Write an expression for the **volume** of a slice of the submarine at position x and of thickness Δx. Include units.

Solution: The radius of such a slice is given by $r(x) = \sqrt{x+1}$, so the volume is $\pi(r(x))^2\Delta x = \pi(x+1)\Delta x$ m^3.

b. [2 points] Write an expression for the **mass** of the slice you found in part (a). Include units.

Solution: The density function $p(x)$ depends only on x, so the density is roughly constant on the slice from part (a), as long as Δx is very small. The mass of such a slice is then

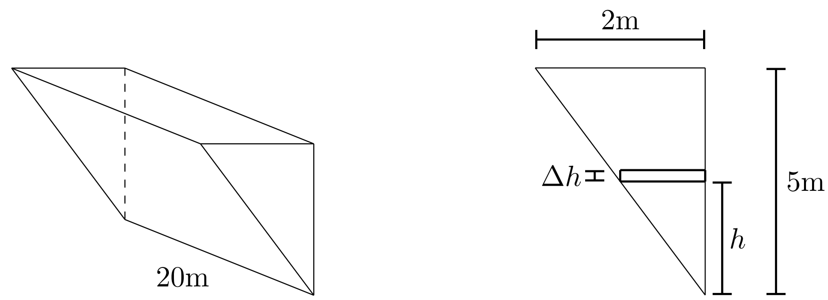
$$M(x) = p(x) \cdot \pi(x+1)\Delta x = [(x-5)^2 + 1]\pi(x+1)\Delta x \text{ kg}.$$

c. [3 points] Write, but do not evaluate, an integral which gives the **total mass** of the submarine. Include units.

Solution: The approximate mass of the submarine is obtained by adding together all the masses of the slices calculated above to get $\sum[(x-5)^2+1]\pi(x+1)\Delta x$. In the limit we get the exact mass in the form of an integral:

$$\int_0^{10}[(x-5)^2+1]\pi(x+1)\,dx.$$

7. [13 points] A drinking water facility needs to pump water out of an underground tank. The tank is 20 meters in length with right-triangular cross-sections perpendicular to the ground as shown in the figure. The top of the tank is a 2m by 20m rectangle. The **top** of the tank lies **5 meters below the surface of the earth.** Recall that $g = 9.8\text{m/s}^2$, where g is the gravitational constant.



Underground Tank

a. [5 points] Write an expression for the **volume** (in cubic meters) of a horizontal rectangular slice of water at height h above the bottom of the tank, with thickness Δh. Your answer should not involve an integral.

Solution: The length of the slice is 20m. Call the width w. To find w in terms of h, we use similar triangles (using the diagram on the right) to set up the proportion:

$$\frac{w}{h} = \frac{2}{5} \quad \Longrightarrow \quad w = \frac{2}{5}h.$$

Therefore the volume of such a slice is (we use that the volume of a rectangular prism is its length times its width times its height):

$$20 \cdot \frac{2}{5}h \cdot \Delta h = 8h\Delta h.$$

b. [2 points] The density of water is approximately 1000 kg/m^3. Write an expression for the **weight** (in Newtons) of the slice of water from part (a). Your answer should not involve an integral.

Solution: The density is constant, and therefore the mass of such a slice is the volume of that slice times 1000 kg/m^3, which is $8000h\Delta h$. To obtain the weight in Newtons, we multiply by $g = 9.8\text{m/s}^2$ to get $9.8 \cdot 8000h\Delta h$.

c. [3 points] Write an expression for the **work** (in Joules) needed to pump the slice of water (from parts (a) and (b)) to the surface of the earth. Your answer should not involve an integral.

Solution: (Note: the wording of this problem was slightly edited for clarity) The slice lies $5-h$ meters below the top of the tank, and the top of the tank is 5 meters below the surface of the earth, so the total distance we move the slice up is $10-h$ meters. Therefore the work (in Joules) done to move one slice up to the surface of the earth is approximately

$$(10-h)\cdot(9.8\cdot 8000h\Delta h).$$

d. [3 points] Assuming the tank is initially full of water, write an integral for the **total work** (in Joules) needed to pump all of the water to the surface of the earth.

Solution: Adding up the contributions of the work needed to move each slice found in part (c) and taking a limit as the thickness Δh of each slice goes to zero, we obtain the exact answer in the form of the integral

$$\int_0^5 (10-h)(9.8\cdot 8000h)\,dh.$$

8. [12 points]

a. [6 points] Split the function

$$f(x) = \frac{x+2}{(x-2)^2(x-1)}$$

into partial fractions. Do not integrate your result. Please show all of your work.

Solution: Start by splitting:

$$\frac{x+2}{(x-2)^2(x-1)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

By multiplying through to get a common denominator, we get

$$x + 2 = A(x-2)^2 + B(x-1)(x-2) + C(x-1). \quad (1)$$

Method 1 (Comparing coefficients): we multiply out the products on the right hand side and group terms which have the same power of x in them. This gives:

$$x + 2 = (A+B)x^2 + (-4A - 3B + C)x + (4A + 2B - C). \quad (2)$$

This gives us the system of equations:

$$A + B = 0, \quad -4A - 3B + C = 1, \quad 4A + 2B - C = 2.$$

We solve this system to obtain values: $A = 3, B = -3, C = 4$.

Method 2 (Plugging in values): If we plug $x = 2$ into (1) we get

$$2 + 2 = A(2-2)^2 + B(2-1)(2-2) + C(2-1),$$

which simplifies to $4 = C$.
If we plug $x = 1$ into (1) we get

$$3 = A(1-2)^2 + B(1-1)(1-2) + C(1-1)$$

which simplifies to $3 = A$.
If we plug these values for A and C back into (1) and also plug in $x = 3$ we obtain the equation

$$\begin{aligned} 3 + 2 &= 3(3-2)^2 + B(3-1)(3-2) + 4(3-1) \\ 5 &= 3 + 2B + 8 \\ -6 &= 2B \\ B &= -3. \end{aligned}$$

So we find $A = 3, B = -3, C = 4$.

8. (continued)

b. [6 points] Given the partial fraction decomposition

$$\frac{-3x}{(x+1)(x^2+1)} = \frac{3}{2(x+1)} - \frac{3(x+1)}{2(x^2+1)},$$

evaluate the following indefinite integral, showing all of your work:

$$\int \frac{-3x}{(x+1)(x^2+1)}\,dx.$$

Solution: Start by splitting up the integral:

$$\int \frac{-3x}{(x+1)(x^2+1)}\,dx = \int \frac{3}{2(x+1)}\,dx - \int \frac{3(x+1)}{2(x^2+1)}\,dx.$$

Then we split up the second integral to get

$$\int \frac{-3x}{(x+1)(x^2+1)}\,dx = \int \frac{3}{2(x+1)}\,dx - \int \frac{3x}{2(x^2+1)}\,dx - \int \frac{3}{2(x^2+1)}\,dx.$$

For the first integral, we have:

$$\int \frac{3}{2(x+1)}\,dx = \frac{3}{2}\int \frac{1}{x+1}\,dx = \frac{3}{2}\ln|x+1| + C.$$

For the second integral, we u-substitution with $u = x^2+1$, and $du = 2xdx$, so:

$$\int \frac{3x}{2(x^2+1)}\,dx = \frac{3}{2}\int \frac{x}{x^2+1}\,dx = \frac{3}{4}\int \frac{1}{u}\,du = \frac{3}{4}\ln|x^2+1| + C.$$

For the final integral, we have:

$$\int \frac{3}{2(x^2+1)}\,dx = \frac{3}{2}\int \frac{1}{x^2+1}\,dx = \frac{3}{2}\arctan(x) + C.$$

Putting this all together, we get

$$\int \frac{-3x}{(x+1)(x^2+1)}\,dx = \frac{3}{2}\ln|x+1| - \frac{3}{4}\ln|x^2+1| - \frac{3}{2}\arctan(x) + C.$$

9. [10 points]

a. [6 points] As a particle moves in the xy plane, it traces out the curve $y = g(x)$ for $1 \leq x \leq 5$, where $g(x)$ is the function

$$g(x) = \int_{2x}^{x^4} \sin(t^3)\, dt + 100e^{\pi}.$$

Set up, but do not evaluate, an expression with a *single* integral which gives the arclength of the path of the particle. Your answer should not involve the letter g.

Solution: The arclength formula says that the length of the path of the particle is given by

$$\int_1^5 \sqrt{1 + (g'(x))^2}\, dx.$$

The FTC gives

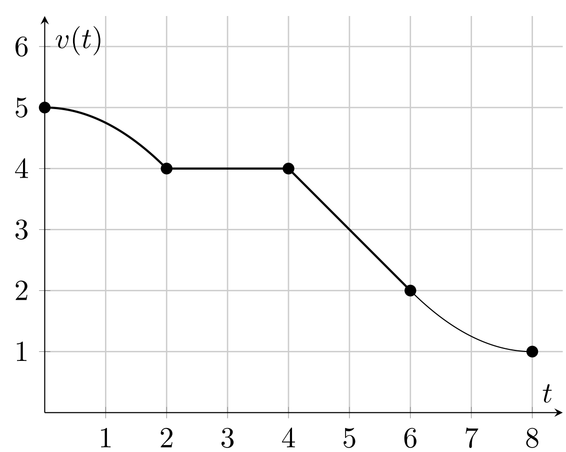
$$g'(x) = 4x^3 \sin(x^{12}) - 2\sin(8x^3).$$

Therefore, the answer is

$$\int_1^5 \sqrt{1 + \left[4x^3 \sin(x^{12}) - 2\sin(8x^3)\right]^2}\, dx.$$

b. [4 points] A different particle is traveling with velocity given by $v(t)$ meters/second, where $v(t)$ is the function in the following graph: The units of t are seconds.

Graph of $y = v(t)$



Using a left Riemann sum with four subdivisions, estimate the distance the particle travels in the first 8 seconds of its journey. Is this an underestimate or an overestimate of the actual distance traveled in the first 8 seconds? Justify your answer.

Solution: Since the length of each subdivision is 2, we obtain:

$$\begin{aligned}\text{LEFT}(4) &= 2\cdot 5 + 2\cdot 4 + 2\cdot 4 + 2\cdot 2\\ &= 2(5+4+4+2)\\ &= 30.\end{aligned}$$

This is an overestimate of the actual distance. This is because the function $v(t)$ is decreasing on $[0, 2]$ and $[4, 8]$ and constant on $[2, 4]$. Left Riemann sums overestimate the actual area for decreasing functions.