Math 116 — Second Midterm — November 15, 2022

Write your 8-digit UMID number very clearly in the box to the right.

Your Initials Only: ______ Instructor Name: ____________________ Section #: ____

1. This exam has 16 pages including this cover.
2. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. You are allowed notes written on two sides of a $3'' \times 5''$ note card.
6. You are NOT allowed other resources, including, but not limited to, notes, calculators or other devices.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	8	
3	15	
4	9	
5	12	
6	11	

Problem	Points	Score
7	7	
8	5	
9	8	
10	14	
Total	100	

1. [11 points]

a. [7 points] Determine the **radius** of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{9^n (x-2)^{2n}}{n^2}$$

Be sure to show all of your work. Write your final answer in the space provided below.

Solution: We use the ratio test, with $a_n = \frac{9^n(x-2)^{2n}}{n^2}$. Then:

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} &= \frac{9^{n+1}|x-2|^{2n+2}n^2}{(n+1)^2 9^n |x-2|^{2n}} \\ &= 9|x-2|^2 \frac{n^2}{(n+1)^2} \\ &\to 9|x-2|^2 \quad \text{as } n \to \infty.\end{aligned}$$

This is less than 1 exactly when $9|x-2|^2 < 1$, or in other words $|x-2| < 1/3$. So the radius of convergence is $1/3$.

b. [4 points] Suppose that the power series

$$\sum_{n=1}^{\infty} a_n (x-5)^n$$

Answer: $1/3$.

converges when $x = 10$ and diverges when $x = -1$. At which of the following x-values must the series converge? Circle your answers. You do not need to show any work for this problem.

-5 $\quad$ 0 $\quad$ $\boxed{2}$ $\quad$ $\boxed{5}$ $\quad$ 11 $\quad$ 12

Solution: From the information given, the radius of convergence is at least 5 and at most 6. Hence it definitely converges for $|x-5|<5$, but we don't know if it converges when $|x-5| \geq 5$. So it definitely converges at $x = 2, 5$.

2. [8 points] Suppose that a_n, b_n, and c_n are sequences with the following properties:

- The sequence a_n is bounded
- The series $\sum_{n=1}^{\infty} b_n$ converges absolutely
- $\frac{1}{n^2+1} \leq c_n \leq \frac{1}{n}$ for all $n \geq 1$

Determine whether the following statements are **always**, **sometimes**, or **never** true, and circle the appropriate answer for each part. No justification is necessary.

a. [2 points] The sequence b_n converges to 0.

Circle one: $\boxed{\textbf{Always}}$ **Sometimes** **Never**

Solution: This is always true by the nth term test for divergence.

b. [2 points] $\sum_{n=1}^{\infty} \frac{c_n}{n}$ diverges.

Circle one: **Always** **Sometimes** $\boxed{\textbf{Never}}$

Solution: This is never true by the comparison test, with comparison series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

c. [2 points] The sequence a_n converges.

Circle one: **Always** $\boxed{\textbf{Sometimes}}$ **Never**

Solution: This is sometimes true. For example, the sequence $a_n = 0$ for all $n \geq 1$ is bounded and converges; on the other hand, the sequence $a_n = (-1)^n$ for $n \geq 1$ is bounded and does not converge.

d. [2 points] The series $\sum_{n=1}^{\infty} \frac{1}{n^3 c_n}$ converges.

Circle one: **Always** $\boxed{\textbf{Sometimes}}$ **Never**

Solution: This is sometimes true. By multiplying through by n^3 and then taking reciprocals of the inequality that c_n satisfies, we obtain

$$\frac{1}{n^2} = \frac{n}{n^3} \leq \frac{1}{n^3 c_n} \leq \frac{n^2+1}{n^3}.$$

If $\frac{1}{n^3 c_n} = \frac{1}{n^2}$, then this is an example where the corresponding series converges; on the other hand, if $\frac{1}{n^3 c_n} = \frac{n^2+1}{n^3}$, this is an example where the corresponding series diverges.

3. [15 points] Carlos and Nancy are catching a train that leaves at 4pm. They leave their apartment for the train station at 12pm. The amount of time t (in hours) that elapses between the time they leave their apartment and the time they arrive at the train station is described by the following **probability density function** (pdf) $h(t)$:

$$h(t) = \begin{cases} 0 & t \leq 3 \\ a(t-3) & 3 < t \leq 4 \\ \frac{1}{4}e^{4-t} & 4 < t < \infty. \end{cases}$$

a. [5 points] What is the probability they arrive late for their train (i.e., what is the probability they arrive at the train station after 4pm)? Be sure to show work for your calculations, and be sure to use proper notation.

Solution: The probability they arrive after 4pm is given by

$$\int_4^\infty h(t)\,dt.$$

Substituting the formula given for $h(t)$ on this interval yields the improper integral

$$\int_4^\infty \frac{1}{4}e^{4-t}\,dt.$$

We evaluate this:

$$\begin{aligned} \int_4^\infty \frac{1}{4}e^{4-t}\,dt &= \lim_{b\to\infty}\int_4^b \frac{1}{4}e^{4-t}\,dt \\ &= \lim_{b\to\infty}\left(-\frac{1}{4}e^{4-t}\right)\Big|_4^b \\ &= \lim_{b\to\infty}\left(\frac{1}{4}-\frac{1}{4}e^{4-b}\right) \\ &= \frac{1}{4}. \end{aligned}$$

So the answer is $1/4$.

b. [4 points] Find the value of a so that $h(t)$ is a probability density function. Be sure to show work for any calculations.

Solution: Since $h(t)$ is a pdf, its integral from $-\infty$ to ∞ evaluates to 1. We already found $\int_4^\infty \frac{1}{4}e^{4-t}\,dt = 1/4$, so

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} h(t)\,dt \\ &= \int_3^4 a(t-3)\,dt + \int_4^\infty \frac{1}{4}e^{4-t}\,dt \\ &= \int_3^4 a(t-3)\,dt + \frac{1}{4}. \end{aligned}$$

We evaluate the first integral:

$$\begin{aligned} \int_3^4 a(t-3)\,dt &= \frac{at^2}{2}\Big|_3^4 - 3at\Big|_3^4 \\ &= 8a - \frac{9a}{2} - 3a \\ &= \frac{a}{2}. \end{aligned}$$

Therefore we have

$$\frac{a}{2} = 1 - \frac{1}{4} = \frac{3}{4},$$

so $a = 3/2$.

c. [3 points] Give a practical interpretation of the fact that $h(4.5) = 0.15$.
Note that the output value has been rounded to the nearest hundredth.

Solution: The probability that Carlos and Nancy arrive between 3:59 and 4:01 (this is a time interval of 2 minutes, or 1/30 hour) is approximately $\frac{1}{30} \cdot 0.15$.

d. [3 points] Write an expression involving one or more integrals that gives the mean amount of time it takes Nancy and Carlos to travel to the train station. The letter h should not appear in your answer. You do not need to evaluate any integrals for this part.

Solution: The mean is

$$\int_{-\infty}^{\infty} th(t)\, dt = \int_3^4 \frac{3}{2}t(t-3)\, dt + \int_4^{\infty} \frac{1}{4}e^{4-t}\, dt.$$

4. [9 points] Consider the following sequences, all defined for $n = 1, 2, 3, \ldots$

$$a_n = \int_1^n \frac{1}{x^2}\,dx$$
$$b_n = 1 + \frac{(-1)^n(n+1)}{n^2}$$
$$c_n = \sum_{k=1}^{n} \frac{1}{(k+1)^{0.5}}$$

a. [3 points] Which sequences are monotone? No justification is required for this part of the problem. Circle your final answer(s) below.

Circle your answers: $\boxed{a_n}$ b_n $\boxed{c_n}$ NONE

b. [3 points] Which sequences are convergent? No justification is required for this part of the problem. Circle your final answer(s) below.

Circle your answers: $\boxed{a_n}$ $\boxed{b_n}$ c_n NONE

c. [3 points] Does the series $\sum_{n=1}^{\infty} a_n$ converge? Justify your answer.

Solution: Note that

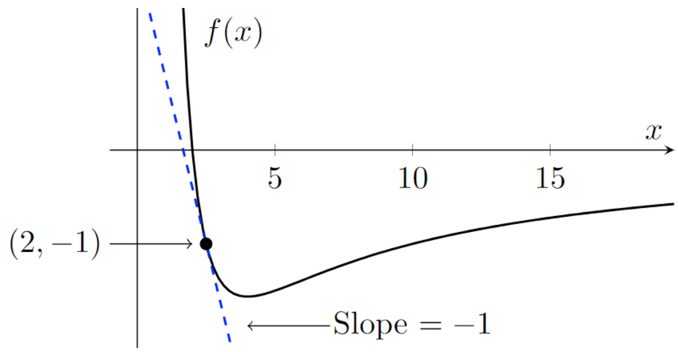
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \int_1^n \frac{1}{x^2}\,dx = \int_1^\infty \frac{1}{x^2}\,dx.$$

This integral converges to a nonzero number. By the nth term test for divergence, the series $\sum_{n=1}^{\infty} a_n$ diverges.

5. [12 points] The solid curve graphed below is part of the graph of a function $f(x)$ which has the following properties:

- $f(x)$ is twice differentiable on the interval $(0, \infty)$.
- $f(2) = -1$.
- For all $x \geq 10$, $f(x) < -\dfrac{5}{x}$.

The dashed line is the tangent line to $f(x)$ at $x = 2$, and its slope is -1.



a. [3 points] Compute $\displaystyle\lim_{x\to 2}\left(\frac{f(x)+1}{\cos\left(\frac{\pi}{2}x\right)+\frac{1}{2}x}\right)$

Solution: Note that since $f(x)$ is differentiable, hence continuous,

$$\lim_{x\to 2} f(x) + 1 = f(2) + 1 = 0.$$

Also, the limit of $\cos(\pi \cdot x/2) + x/2$ as $x \to 2$ is 0, so we apply L'Hôpital's rule:

$$\begin{aligned}\lim_{x\to 2}\frac{f(x)+1}{\cos\left(\frac{\pi}{2}x\right)+\frac{x}{2}} &\overset{LH}{=} \lim_{x\to 2}\frac{f'(x)}{-\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)+\frac{1}{2}} \\ &= \frac{f'(2)}{1/2} \\ &= -2.\end{aligned}$$

b. [3 points] Compute $\displaystyle\lim_{x\to\infty} x[f(2+x^{-1})+1]$

Solution: Since the limit as x tends to infinity of $f(2+x^{-1})$ is $f(2) = -1$, this is an indeterminate limit of the form $\infty \cdot 0$, so L'Hôpital's rule applies. We get:

$$\begin{aligned}\lim_{x\to\infty} x[f(2+x^{-1})+1] &= \lim_{x\to\infty}\frac{f(2+x^{-1})+1}{x^{-1}} \\ &\overset{LH}{=} \lim_{x\to\infty}\frac{-x^{-2}f'(2+x^{-1})}{-x^{-2}} \\ &= f'(2) \\ &= -1.\end{aligned}$$

c. [6 points] Does the following improper integral converge or diverge? Fully justify your answer including using proper notation and showing mechanics of any tests you use.

$$\int_1^\infty (-f(x))\,dx.$$

Solution: This diverges by the comparison test. It suffices to check that $\int_{10}^\infty f(x)\,dx$ diverges. For this, we note that the inequality $f(x) < -5/x$ implies that

$$-f(x) > \frac{5}{x},$$

since $A < B$ implies that $-A > -B$. Furthermore, $\int_{10}^\infty \frac{5}{x}\,dx$ diverges by the p-test with $p = 1$. By the comparison test, $\int_{10}^\infty (-f(x))\,dx$ diverges as well.

6. [11 points] It's fall, and the leaves are falling onto the grass. Over the course of each day, suppose that 10kg of leaves fall onto the grass. At the beginning of each day, 50 percent of the leaves currently on the grass (from previous days) are removed.

a. [3 points] Let M_n be the total mass in kg of all the leaves currently on the grass at the **end** of the nth day. Suppose that $M_1 = 10$. Write expressions for M_2 and M_3. The letter M should not appear in your answer.

Solution: $M_2 = \frac{1}{2} \cdot 10 + 10 = 15$, $M_3 = \frac{1}{2} \cdot 15 + 10 = 17.5$.

b. [5 points] Find a closed-form expression for M_n. This means that your answer should be a function of n, should not contain Σ, and should not be recursive.

Solution: Note that

$$\begin{aligned}
M_1 &= 10 \\
M_2 &= 10 + 10 \cdot \frac{1}{2} \\
M_3 &= 10 + \frac{1}{2}\left(10 + 10 \cdot \frac{1}{2}\right) = 10 + 10 \cdot \frac{1}{2} + 10 \cdot \left(\frac{1}{2}\right)^2 \\
&\vdots \\
M_n &= 10 + 10 \cdot \frac{1}{2} + 10 \cdot \left(\frac{1}{2}\right)^2 + \cdots + 10 \cdot \left(\frac{1}{2}\right)^{n-1} = \sum_{k=0}^{n-1} 10\left(\frac{1}{2}\right)^k.
\end{aligned}$$

Therefore

$$M_n = 10\frac{1 - (1/2)^n}{1 - 1/2} = 20(1 - (1/2)^n).$$

c. [3 points] If 30kg of leaves are on the grass at the end of any day, then the grass will die. Will this happen during the fall? Justify your answer.

Solution: The long-term mass in kg is found by taking the limit of our answer from part (b), and is

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} 10(1 - (1/2)^n) = 20.$$

Since $20 < 30$, and since the sequence M_n is monotonically increasing, this means that $M_n \le 20 < 30$ for all n. Therefore, there will never be 30 or more kg of leaves on the grass, so the grass will not die.

7. [7 points] Determine whether the following improper integral converges or diverges. Circle your final answer choice. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

$$\int_1^\infty \frac{t^2 - \ln(t)}{t^4 + 8t + 10}\, dt.$$

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Solution: The numerator is dominated by t^2, and the denominator is dominated by t^4, so the integrand has the same behavior (for large t) as $\frac{t^2}{t^4} = \frac{1}{t^2}$, whose integral on the interval $[1, \infty)$ converges. Therefore we expect that this improper integral converges. To show this, first note that since $\ln(t) \geq 0$ for $t \geq 1$, we have

$$t^2 - \ln(t) \leq t^2.$$

Also, since $8t + 10 \geq 0$ for $t \geq 1$, we have

$$t^4 + 8t + 10 \geq t^4.$$

Therefore

$$\frac{t^2 - \ln(t)}{t^4 + 8t + 10} \leq \frac{t^2}{t^4} = \frac{1}{t^2}.$$

Now, $\int_1^\infty \frac{1}{t^2}\, dt$ converges by the p-test with $p = 2$. Hence, by the comparison test, our integral converges as well.

8. [5 points] Fully evaluate the following integral:

$$\int x \ln x \, dx$$

You do not need to simplify your answer.

Solution: This integral can be done via integration by parts in two ways.
Method 1. Let $u = x, dv = \ln(x)\,dx$, so $du = dx, v = x\ln(x) - x$. Then integration by parts gives

$$\int x\ln(x)\,dx = -\int x\ln(x)\,dx + \int x\,dx + x(x\ln(x) - x).$$

Adding $\int x\ln(x)\,dx$ to both sides and performing the antidifferentiation on the right-hand side gives

$$2\int x\ln(x)\,dx = -\frac{1}{2}x^2 + x^2\ln(x) + C.$$

We then solve to get

$$\int x\ln(x)\,dx = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C.$$

Method 2. Let $u = \ln(x), dv = x\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{x^2}{2}$. Integration by parts gives

$$\begin{aligned}\int x\ln(x)\,dx &= -\int \frac{x^2}{2}\cdot\frac{1}{x}\,dx + \ln(x)\cdot\frac{x^2}{2}\\ &= -\int \frac{1}{2}x\,dx + \frac{x^2}{2}\ln(x)\\ &= \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C.\end{aligned}$$

9. [8 points] For each of the following, circle ALL that apply. There may be more than one correct answer for each part. You do not need to show any work for any part of this question.

a. [4 points] Suppose $P(t)$ is a **cumulative distribution function** (cdf) satisfying $P(0.2) = 0.5$. Which of the following MUST be true?

$\boxed{\lim_{t\to\infty} P(t) = 1}$ $\boxed{\text{The median } t\text{-value is } 0.2.}$

$\boxed{P(1) \geq 0.5}$ The mean t-value is 0.2.

NONE OF THESE

b. [4 points] The series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n + \sin(n)}{n^{3.1}}$...

DIVERGES   CONVERGES CONDITIONALLY

$\boxed{\text{CONVERGES}}$ $\boxed{\text{CONVERGES ABSOLUTELY}}$

NONE OF THESE

Solution: (The following justification was not required to get points on this problem.) Note that

$$\frac{n + \sin(n)}{n^{3.1}} \leq \frac{n+1}{n^{3.1}},$$

since $-1 \leq \sin(n) \leq 1$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^{3.1}}$ converges by limit comparison test, with comparison series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2.1}}$. This converges by the p-test with $p = 2.1$. Therefore our original series converges absolutely by the comparison test. Since it converges absolutely, it also converges.

10. [14 points] Determine if the following series converge or diverge. Circle your final answer choice for each. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

a. [7 points] $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+4n-1}$

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Solution: With $a_n = \frac{n}{n^2+4n-1}$, we see that $a_n > a_{n+1}$ for all $n \geq 1$. Furthermore, $\lim\limits_{n\to\infty} a_n = 0$. By the alternating series test, our original series converges.

b. [7 points] $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{n^2+n+2}$

Circle one: **Converges** $\boxed{\textbf{Diverges}}$

Solution: By leading term analysis, this series has terms reminiscent of $\frac{2n}{n^2} = \frac{2}{n}$. Since $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges by the p-test with $p = 1$, we expect divergence. To justify this we use LCT. Let $b_n = \frac{2}{n}$ and let $a_n = \frac{2n-1}{n^2+n+2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2-n}{2n^2+2n+4} = 1.$$

Since $0 < 1 < \infty$, LCT implies that the given series and our comparison series have the same behavior. Therefore our original series diverges.