Math 116 — Final Exam — December 14, 2022

Write your 8-digit UMID number very clearly in the box to the right.

Your Initials Only: ______ Instructor Name: ____________________________ Section #: ____

1. This exam has 17 pages including this cover.
2. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. You are allowed notes written on two sides of a $3'' \times 5''$ note card.
6. You are NOT allowed other resources, including, but not limited to, notes, calculators or other devices.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	15	
3	13	
4	12	
5	11	
6	13	

Problem	Points	Score
7	12	
8	13	
Total	100	

1. [11 points] Nancy is on an airplane traveling to see Carlos, and the flight is delayed. The function $f(t)$ is the probability density function (pdf) for how many hours, t, the flight will be delayed. The function $f(t)$ is given by

$$f(t) = \begin{cases} 0 & \text{if } t \le 0, \\ -\frac{3}{4}(t-1)^2 + \frac{3}{4} & \text{if } 0 < t < 2, \\ 0 & \text{if } t \ge 2 \end{cases}$$

a. [6 points] Write a formula for the cumulative distribution function (cdf) $F(t)$ corresponding to the pdf $f(t)$. Your answer should not involve any integrals or the letter $f(t)$. Write your answer using the partially given piecewise notation below.

Solution: The cdf is given by

$$\begin{aligned} F(t) &= \int_{-\infty}^{t} f(s)\,ds \\ &= \int_{0}^{t} f(s)\,ds. \end{aligned}$$

The second equality is true because $f(t) = 0$ for $t \le 0$. This also means that $F(t) = 0$ for $t \le 0$. If $0 < t < 2$, then we have:

$$\begin{aligned} F(t) &= \int_{0}^{t} f(s)\,ds \\ &= \int_{0}^{t} \Big(-\frac{3}{4}(s-1)^2 + \frac{3}{4} \Big)\,ds \\ &= \Big(-\frac{1}{4}(s-1)^3 + \frac{3}{4}s \Big)\Big|_0^t \\ &= -\frac{1}{4}(t-1)^3 - \frac{1}{4} + \frac{3}{4}t. \end{aligned}$$

This gives $F(t)$ for $0 < t < 2$. For $t \ge 2$, note that $f(t) = 0$ for all such t, and therefore $F(t)$ must be constant on $(2, \infty)$. Since cdfs must tend to 1 as the input tends to ∞, this means $F(t) = 1$ for $t \ge 2$. We can also see this by computing: if $t \ge 2$,

$$\begin{aligned} F(t) &= \int_{0}^{t} f(s)\,ds \\ &= \int_{0}^{2} f(s)\,ds + \int_{2}^{t} f(s)\,ds \\ &= \int_{0}^{2} \Big(-\frac{3}{4}(s-1)^2 + \frac{3}{4} \Big)\,ds \\ &= \Big(-\frac{1}{4}(s-1)^3 + \frac{3}{4}s \Big)\Big|_0^2 \\ &= -\frac{1}{4} + \frac{3}{2} - \frac{1}{4} = 1. \end{aligned}$$

Below we write down the formula for $F(t)$ using the provided piecewise notation.

$$F(t) = \begin{cases} 0 & \textbf{if } t \le 0 \\ -\frac{1}{4}(t-1)^3 + \frac{3}{4}t - \frac{1}{4} & \textbf{if } 0 < t < 2 \\ 1 & \textbf{if } t \ge 2 \end{cases}$$

b. [2 points] What is the probability that Nancy's flight will be delayed less than 30 minutes?

Solution: This is equal $F(1/2)$. We can use our answer to part (a) to get

$$F(1/2) = -\frac{1}{4}(-1/2)^3 + \frac{3}{4} \cdot \frac{1}{2} - \frac{1}{4} = \frac{5}{32}.$$

c. [3 points] Carlos wants to find the mean amount of time the flight will be delayed, so he can arrive at the airport at the right time. Write an explicit expression involving integrals that gives the mean amount of time the flight will be delayed. Do not evaluate your expression. Your answer should not contain the letter f.

Solution: The mean is:

$$\int_{-\infty}^{\infty} tf(t)\,dt = \int_0^2 \left(-\frac{3}{4}t(t-1)^2 + \frac{3}{4}t \right) dt$$

2. [15 points] The parts of this problem are unrelated to each other. Be sure to show work for all parts, and circle your final answer.

a. [5 points] A leaking bag of sugar is lifted vertically from the ground to a height of 10 feet above the ground. The **weight** of the bag of sugar is $6 - \sqrt{x}$ lbs when it has been lifted x feet above the ground. Find the work done lifting the bag, including units. Fully evaluate any integrals, but you do not need to simplify your answer.

Solution: The work is obtained by integrating the force over the distance the bag is lifted. The force on the bag is equal to its weight, so we have:

$$\int_0^{10} (6 - \sqrt{x})\, dx = 60 - \frac{2}{3}x^{3/2}\Big|_0^{10}$$
$$= 60 - \frac{2}{3}10^{3/2}.$$

Answer: $60 - \frac{2}{3}10^{3/2}$ **lbs·ft**

b. [5 points] Write an expression involving one or more integrals that gives the volume of the solid obtained by rotating the region in the xy-plane bounded between the x-axis, the parabola $y = x^2 + 1$, the line $x = -1$ and the line $x = 1$, about the line $x = -2$. Do not evaluate your integral(s).

Solution: Using the shell method, the volume is

$$\int_{-1}^{1} 2\pi(x+2)(x^2+1)\, dx.$$

Answer: $\int_{-1}^{1} 2\pi(x+2)(x^2+1)\, dx$

c. [5 points] The function $f(x) = x^4 + 5$ can be rewritten in the form $f(x) = (x+1)^4 + A(x+1)^3 + B(x+1)^2 + C(x+1) + D$, where A, B, C, D are constants. Find the values of A, B, C, D using Taylor series. Other methods used to find the constants will not be given credit.

$A=$ -4

$B=$ 6

$C=$ -4

$D=$ 6

3. [13 points] A function $g(x)$ has Taylor series centered at $x = 5$ given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^{n+1}}{(n+1) \cdot 4^n}.$$

a. [2 points] Is $g(x)$ increasing or decreasing near $x = 5$? Briefly justify your answer.

Solution: The coefficient of $(x-5)$ in the Taylor series is equal to $g'(5)$. Therefore $g'(5) = \frac{(-1)^0}{(0+1) \cdot 4^0} = 1 > 0$, so $g(x)$ is increasing near $x = 5$.

b. [3 points] Find $g^{(1001)}(5)$.

Solution: The coefficient of $(x-5)^n$ in the Taylor series for $g(x)$ is $\frac{g^{(n)}(5)}{n!}$. We see that the exponent on $(x-5)$ is 1001 when $n = 1000$. Therefore $g^{(1001)}(5)$ is equal to 1001! times $\frac{(-1)^{1000}}{(1000+1) \cdot 4^{1000}}$.

$g^{(1001)}(5) =$ $\frac{1001!}{1001 \cdot 4^{1000}}$

c. [8 points] Given that the radius of convergence of this Taylor series is 4 (do NOT show this), find the **interval** of convergence of this Taylor series. Show all your work, including full justification for series behavior.

Solution: Since we know the radius of convergence, we just need to test the behavior at the endpoints, which are $5 \pm 4 = 1, 9$. At $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-4)^{n+1}}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{4(-1)^n(-1)^{n+1}}{n+1} = -4\sum_{n=0}^{\infty} \frac{1}{n+1}.$$

To determine the behavior of this, we use the limit comparison test with comparison series $\sum_{n=1}^{\infty} \frac{1}{n}$. We have (with $a_n = \frac{1}{n+1}$ and $b_n = \frac{1}{n}$):

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Since 1 is a positive number, and since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series test with $p = 1$, the limit comparison tells us that the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges. Therefore $-4\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges. So $x = 1$ is not included in the interval of convergence.

At $x = 9$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^{n+1}}{(n+1)4^n} = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

Since $a_n = \frac{1}{n+1} \to 0$ as $n \to \infty$, and a_n is decreasing, by the alternating series test this series converges.

Interval of convergence: $(1, 9]$

4. [12 points] Katya and Miles are sailing in the ocean, which is represented by the xy-plane. Katya's position, t hours after 12:00pm, is given by

$$x = 3t, \quad y = \sin\left(\frac{\pi t}{2}\right),$$

while Miles' position, t hours after 12:00pm, is given by

$$x = t^2 + 2, \quad y = \cos(\pi t) - 1.$$

In this problem, x and y have units in kilometers. All above equations are valid for $0 \leq t \leq 6$.

a. [2 points] What is **Miles'** position at 3:00pm?

Solution: To find this, we plug $t = 3$ into the equations given for the x and y coordinates of Miles.

$x =$ **11** $\qquad y =$ -2

b. [4 points] Will Katya and Miles ever collide during their journey? If so, at what time(s) will this occur? Justify your answer.

Solution: We first consider when their x-coordinates will be equal:

$$\begin{aligned} 3t &= t^2 + 2 \\ t^2 - 3t + 2 &= 0 \\ (t-2)(t-1) &= 0 \end{aligned}$$

has solutions at $t = 1, 2$. We test their y-coordinates at each of these times. At $t = 1$, Katya's y-coordinate is 1, and Miles' is -2, so they do not collide then. At $t = 2$, Katya's y-coordinate is 0, as is Miles'. Therefore they collide then.

The time(s) is/are **2**

c. [3 points] What is the slope of the tangent line to **Katya's** path at $t = 4$?

Solution: We have $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)}{3}$, which at $t = 4$ evaluates to $\frac{\frac{\pi}{2}(1)}{3} = \frac{\pi}{6}$.

The slope is $\frac{\pi}{6}$

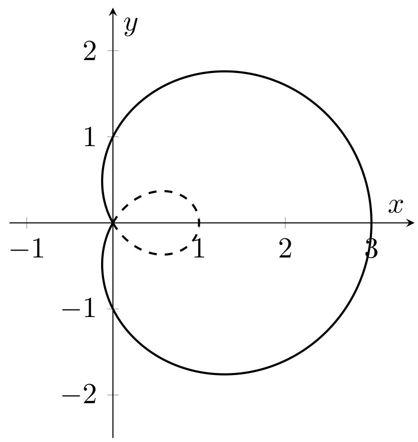
d. [3 points] Write an expression involving one or more integrals that gives the distance, in kilometers, **Miles** traveled between 1:00pm and 4:00pm. Do not evaluate your integral(s).

Solution: For this we use the parametric arc length formula $\int_1^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ with

$$\frac{dx}{dt} = 2t$$
$$\frac{dy}{dt} = -\pi\sin(\pi t).$$

The distance is $\int_1^4 \sqrt{(2t)^2 + (\pi\sin(\pi t))^2}\, dt$

5. [11 points] The parts of this question relate to the following polar graph, defined by the polar curve $r(\theta) = -1 + 2\cos(\theta)$, on the domain $[0, 2\pi]$. Both the solid and dashed curves are part of the graph of $r(\theta)$.



a. [2 points] What are all the angles θ, with $0 \leq \theta \leq 2\pi$, for which the graph passes through the origin?

Solution: The curve passes through the origin when $r(\theta) = 0$, so we need to solve $\cos(\theta) = 1/2$. This occurs for θ in $[0, 2\pi]$ at the values $\theta = \pi/3, 5\pi/3$.

Answer(s): $\theta = \pi/3, 5\pi/3$

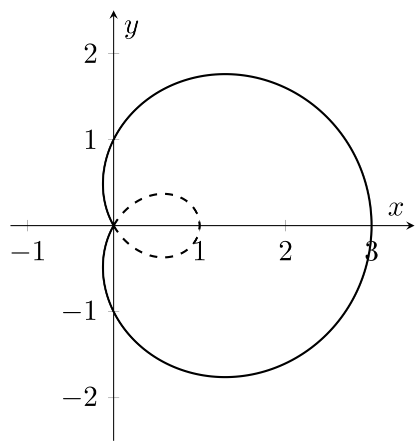
b. [2 points] Determine the interval(s) within $[0, 2\pi]$ for which θ traces out the **dashed** portion of the graph.

Answer(s): $[0, \pi/3]$ **and** $[5\pi/3, 2\pi]$

c. [3 points] Write, but do not evaluate, an expression involving one or more integrals which gives the **area** enclosed by the **dashed** portion of the graph.

The area is $\frac{1}{2}\int_0^{\pi/3}(-1+2\cos(\theta))^2\,d\theta + \frac{1}{2}\int_{5\pi/3}^{2\pi}(-1+2\cos(\theta))^2\,d\theta$

5. (continued) For your convenience, the polar graph referenced by this problem is reproduced here:



d. [4 points] Write, but do not evaluate, an expression involving one or more integrals which gives the **arc length** of the **solid** portion of the graph.

Solution: The arclength of the graph is given by

$$\int_{\pi/3}^{5\pi/3}\sqrt{(r(\theta))^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

We have $dr/d\theta = -2\sin(\theta)$. This gives the answer below.

The arc length is $\int_{\pi/3}^{5\pi/3}\sqrt{(-1+2\cos(\theta))^2+(-2\sin(\theta))^2}\,d\theta$

6. [13 points] Some values of the function $f(x)$, its derivatives, and second derivatives are given in the table below. Assume for all positive integers n that $f^{(n)}(x)$ is continuous for all real numbers x.

x	-2	0	2	4	6
$f(x)$	1	2	0	1	2
$f'(x)$	3	2	1	0	-2
$f''(x)$	-3	-2	0	2	1

Using the information given above, find the following. Be sure to show all of your work. Your answers should not involve the letter f, but you do not need to simplify them.

a. [4 points] Find $\int_{-2}^{2} f'(x)f''(x)\,dx$.

Solution: We do a u-sub with $u = f'(x)$, so $du = f''(x)dx$. Therefore

$$\begin{aligned}\int_{-2}^{2} f'(x)f''(x)\,dx &= \int_{f'(-2)}^{f'(2)} u\,du \\ &= \int_{3}^{1} u\,du \\ &= \frac{u^2}{2}\Big|_3^1 \\ &= \frac{1}{2} - \frac{9}{2} = -4.\end{aligned}$$

Answer: -4

b. [3 points] Find the second degree polynomial that best approximates $f(x)$ near $x = 6$.

Solution: This is the second-degree Taylor polynomial of $f(x)$ near $x = 6$, which is $f(6) + f'(6)(x-6) + \frac{f''(6)}{2!}(x-6)^2$.

$f(x) \approx$ $2 - 2(x-6) + \frac{1}{2}(x-6)^2$

c. [3 points] Find $\lim_{x \to 0} \dfrac{f(x) - 2 + x^2}{x}$

Solution: Since $\lim_{x \to 0} f(x) = f(0)$ (as $f(x)$ is continuous) and $f(0) = 2$, this limit is in indeterminate form $0/0$, so we use L'Hopital's rule:

$$\lim_{x \to 0} \frac{f(x) - 2 + x^2}{x} \stackrel{LH}{=} \lim_{x \to 0} \frac{f'(x) + 2x}{1} = 2.$$

The limit is **2**

d. [3 points] Find the approximate value of $\int_{-2}^{6} x^2 f(x)\, dx$ using MID(2).

Solution: The subintervals we use are $[-2, 2]$ and $[2, 6]$ which have midpoints 0 and 4, respectively. This means:

$$\text{MID}(2) = \Delta x \cdot (0^2 f(0) + 4^2 f(4)) = 4 \cdot (16) = 64.$$

$\int_{-2}^{6} x^2 f(x)\, dx \approx$ **64**

7. [12 points] The rate of vertical growth $r(t)$ of a tree, in meters per **month**, is given by

$$r(t) = \frac{10}{(t+1)^{3/2}}.$$

Here, t is measured in **months** after the tree was planted. **When the tree was planted its height was 1 meter.**

a. [4 points] Write an expression, possibly involving one or more integrals, for the height of the tree after exactly 1 **year** has passed since planting it. You do not need to evaluate your integral(s).

Solution: Let $h(t)$ be the height of the tree, in meters, t months after it was planted. So we are looking for $h(12)$. By the fundamental theorem of calculus,

$$h(12) - h(0) = \int_0^{12} r(t)\, dt.$$

Since at the time it was planted ($t = 0$), the tree was 1 meter tall, $h(0) = 1$. Therefore

$$h(12) = 1 + \int_0^{12} \frac{10}{(t+1)^{3/2}}\, dt.$$

b. [2 points] Let $h(t)$ be the height of the tree, in meters, t **months** after it was planted. Write an expression, possibly involving one or more integrals, for the function $h(t)$. You do not need to evaluate your integral(s).

Solution: By the FTC,

$$h(t) = 1 + \int_0^{t} \frac{10}{(s+1)^{3/2}}\, ds.$$

c. [6 points] Assuming the tree lives long enough, will the tree ever grow more than 20 meters tall? Justify your answer, and be sure to use proper notation.

Solution: The long-term height of the tree, as $t \to \infty$, is

$$\begin{aligned}\lim_{t\to\infty} h(t) &= 1 + \lim_{t\to\infty} \int_0^t \frac{10}{(s+1)^{3/2}}\, ds \\ &= 1 + \lim_{t\to\infty} -20(s+1)^{-1/2}\Big|_0^t \\ &= 1 - 20 \lim_{t\to\infty} \left(\frac{1}{\sqrt{1+t}} - 1\right) \\ &= 21.\end{aligned}$$

Therefore at some point, the tree *will* grow more than 20 meters tall.

8. [13 points] For each part of this problem **circle ALL correct answers**. There may be more than one correct answer for each part. You do not need to show your work.

a. [4 points] Which of the following give a parametrization of the **top half** of the unit circle centered at the origin in the xy-plane?

(A) $x = -\sin(t), \quad y = -\cos(t), \quad \frac{\pi}{2} \le t \le \frac{3\pi}{2}.$

(B) $x = \sin(t), \quad y = \cos(t), \quad \frac{\pi}{2} \le t \le \frac{3\pi}{2}.$

(C) $x = t, \quad y = \sqrt{1-t^2}, \quad -1 \le t \le 1.$

(D) $x = \cos(t), \quad y = \sin(t), \quad \pi \le t \le 2\pi.$

(E) NONE OF THESE

b. [4 points] Which of the following points given in polar coordinates are the same point as $(x, y) = (-1, 1)$ in the xy-plane?

(A) $(r, \theta) = (2, \frac{3\pi}{4})$

(B) $(r, \theta) = (-2, \frac{\pi}{4})$

(C) $(r, \theta) = (\sqrt{2}, -\frac{3\pi}{4})$

(D) $(r, \theta) = (-\sqrt{2}, \frac{7\pi}{4})$

(E) NONE OF THESE

c. [5 points] Which of these options make the following statement true?

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2} + n^2 + n^{3/2}}$ $\quad\ldots$

(A) Diverges by the limit comparison test when compared to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$.

(B) Diverges by the comparison test when compared to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$.

(C) Diverges by the comparison test when compared to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

(D) Converges by the comparison test when compared to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

(E) Converges by the limit comparison test when compared to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(F) Converges because $\frac{1}{n^{1/2}+n^2+n^{3/2}} \to 0$ as $n \to \infty$.

(G) NONE OF THESE

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin\theta$	$\cos\theta$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$