Math 116 — Second Midterm — November 14, 2023

Write your 8-digit UMID number very clearly in the box to the right.

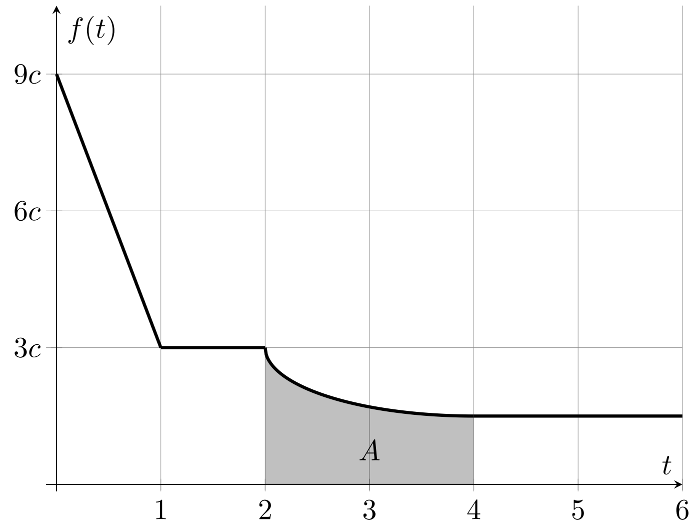
Your Initials Only: ______ Instructor Name: ______________________ Section #: ____

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 12 pages including this cover.
3. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	8	
2	7	
3	12	
4	12	
5	7	
6	12	

Problem	Points	Score
7	18	
8	12	
9	12	
Total	100	

1. [8 points] Ricky's college has installed a new model of napping pod in the library for students to get some well-deserved rest. Let $f(t)$ be the **probability density function** (pdf) for the amount of time, t, Ricky sleeps after he falls asleep in a pod, where t is measured in hours. A **partial** graph of $f(t)$ is shown below. Note that $f(t)$ is piecewise linear on the interval $[0, 2]$ and that $f(t) = 0$ for all $t < 0$.



Additionally, you may assume that c is a positive real number, and that the value of the shaded area between $f(t)$ and the t-axis on $[2, 4]$ is given by the positive number A.

a. [2 points] Suppose $f(8) = 0.03$. Complete the following sentence:

"The probability that Ricky gets between 7.8 hours and 8.2 hours of sleep is ..."

Solution: ... approximately $(0.4)(0.03) = 0.012 = 1.2\%$."

b. [2 points] Find the probability that Ricky gets 1 or fewer hours of sleep. Your answer may be given in terms of c.

Answer: $6c$

c. [2 points] Suppose that there is a 15% chance that Ricky gets 4 hours of sleep or more. Find the value of A in terms of c.

Solution: $9c + A = 0.85$, so $A = 0.85 - 9c$

Answer: $A = 0.85 - 9c$

d. [2 points] The median amount of time Ricky spends sleeping in a pod is 1 hour and 30 minutes. Find the value of c.

Solution: $7.5c = 0.5$, so $c = \frac{0.5}{7.5} = \frac{1}{15}$.

Answer: $\frac{1}{15}$

2. [7 points] Let $f(x)$ and $g(x)$ be two continuous and differentiable functions on $[1, \infty)$. Further, suppose these functions have the following properties:

- $F(x) = \dfrac{g(x)}{x + \ln(x)}$ is an antiderivative of $f(x)$ for $x \geq 1$,
- $g(1) = 11$,
- $\lim\limits_{x\to\infty} g(x) = \infty$,
- $\lim\limits_{x\to\infty} g'(x) = 21$.

Compute the value of the following improper integral if it converges. if it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

$$\int_1^\infty f(x)\,\mathrm{d}x$$

Circle one: **Diverges** $\boxed{\textbf{Converges to } \underline{\quad 10 \quad}}$

Solution: We start by rewriting this improper integral as a limit, and then use the First Fundamental Theorem of Calculus:

$$\begin{aligned}\int_1^\infty f(x)\,\mathrm{d}x &= \lim_{b\to\infty} \int_1^b f(x)\,\mathrm{d}x \\ &= \lim_{b\to\infty} F(b) - F(1) \\ &= \lim_{b\to\infty} \frac{g(b)}{b + \ln(b)} - \frac{g(1)}{1 + \ln(1)} \\ &= \lim_{b\to\infty} \frac{g(b)}{b + \ln(b)} - 11.\end{aligned}$$

As $\lim\limits_{b\to\infty} g(b) = \lim\limits_{b\to\infty} (b + \ln(b)) = \infty$, we try to use L'Hôpital's Rule. We obtain:

$$\begin{aligned}\int_1^\infty f(x)\,\mathrm{d}x &= \lim_{b\to\infty} \frac{g'(b)}{1 + 1/b} - 11 \\ &= \frac{21}{1} - 11 = 21 - 11 = 10.\end{aligned}$$

Therefore, the integral converges to 10.

3. [12 points] Let $p(x)$ be a **probability density function** (pdf) such that

$$p(x) = \begin{cases} 1/10, & -3 \leq x < 1, \\ 3/5, & 1 \leq x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

a. [7 points] Find the cumulative distribution function $P(x)$ corresponding to $p(x)$.

Solution: The continuous function $P(x)$ should approach 0 as $x \to -\infty$, and should be an antiderivative of $p(x)$. We see that $P(x)$ is piecewise linear, and so we use point-slope form for each piece. Note that $P(-3) = 0$ and that, since $\int_{-3}^{1} p(x)\, dx = \frac{2}{5}$, we must have $P(1) = \frac{2}{5}$. We obtain:

$$\textbf{Answer: } P(x) = \begin{cases} 0, & x < -3 \\ \frac{1}{10}(x+3), & -3 \leq x < 1, \\ \frac{3}{5}(x-1) + \frac{2}{5}, & 1 \leq x < 2, \\ 1, & x \geq 2 \end{cases}$$

b. [5 points] Find the mean value of x. Show all your work.

Solution: The mean value is given by:

$$\begin{aligned} \int_{-\infty}^{\infty} x p(x)\, dx &= \int_{-3}^{1} \frac{1}{10} x\, dx + \int_{1}^{2} \frac{3}{5} x\, dx \\ &= \frac{1}{20} x^2 \Big|_{-3}^{1} + \frac{3}{10} x^2 \Big|_{1}^{2} \\ &= \frac{1}{20}(1-9) + \frac{3}{10}(4-1) \\ &= \frac{9}{10} - \frac{4}{10} = \frac{1}{2}. \end{aligned}$$

Answer: 0.5

4. [12 points] The population of fungus-eating insects Megan introduced to stop her mushroom infestation has grown out of control! She decides to spray pesticide to slow their growth. Over the first half of each day, the population of insects increases by 100 percent. Throughout the last half of the day, Megan sprays her garden with insecticide and wipes out exactly k insects. The insects only multiply during the first half of a day, and Megan is only awake to spray the colony during the last half of a day.

a. [5 points] Let B_n denote the number of insects in the colony at the start of the nth day. At the start of the first day, there were 133 insects in the colony (so $B_1 = 133$). Find expressions for the values of B_2, B_3, and B_4. Your answers will involve k but should not involve the letter B. You do not need to simplify your expressions.

$B_2 =$ $2 \cdot 133 - k$

$B_3 =$ $2^2 \cdot 133 - k - 2k$

$B_4 =$ $2^3 \cdot 133 - k - 2k - 4k$

b. [5 points] Find a closed-form expression for B_n. Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive. Your answer will involve k. You do not need to simplify your expression.

Solution: Continuing the pattern above, we see $B_n = 2^{n-1} \cdot 133 - k - 2k - \ldots - 2^{n-2}k$. Aside from the first term in this expression, this is a geometric series with initial term $-k$, common ratio 2, and with $n-1$ terms. Using our formula for a finite geometric series, we obtain:

$$B_n = 2^{n-1} \cdot 133 - k\left(\frac{1-2^{n-1}}{1-2}\right) = 2^{n-1} \cdot 133 - k(2^{n-1}-1).$$

Answer: $2^{n-1} \cdot 133 - k(2^{n-1}-1)$

c. [2 points] Megan can affect the value of k by adjusting the amount of pesticide she sprays. What value must k be so that the population of insects at the start of the nth day is the same for all $n = 1, 2, \ldots$? (Hint: Use your expressions from part **a**.)

Solution: We set $B_1 = B_2$ to get $133 = 2 \cdot 133 - k$. Rearranging gives $k = 133$.

Answer: 133

5. [7 points]
Determine whether the following improper integral converges or diverges and circle the corresponding word. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use. You do not need to calculate the value of the integral if it converges.

$$\int_2^\infty \frac{10 + 3\cos(x^2)}{x^{1/2} + x^{3/2}}\, \mathrm{d}x$$

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Solution: On the interval $x \geq 2$, we have $\dfrac{10+3\cos(x^2)}{x^{1/2}+x^{3/2}} \leq \dfrac{13}{x^{3/2}}$, and $\displaystyle\int_2^\infty \frac{13}{x^{3/2}}\, \mathrm{d}x$ converges by the p-test with $p = \frac{3}{2}$. Therefore, by the (Direct) Comparison Test, $\displaystyle\int_2^\infty \frac{10+3\cos(x^2)}{x^{1/2}+x^{3/2}}\, \mathrm{d}x$ converges.

6. [12 points]

a. [6 points] For each of the following sequences, defined for $n \geq 1$, decide whether the sequence is monotone increasing, monotone decreasing, or neither, and whether it is bounded or unbounded. Circle your answers. No justification is required.

(i) $a_n = \dfrac{\sin(n)}{n}$

Circle all which apply:

Monotone Increasing **Monotone Decreasing** $\boxed{\textbf{Neither}}$

$\boxed{\textbf{Bounded}}$ **Unbounded**

(ii) $b_n = \displaystyle\int_{\frac{1}{n}}^{2} \frac{1}{x^{1/2}}\, dx$

Circle all which apply:

$\boxed{\textbf{Monotone Increasing}}$ **Monotone Decreasing** **Neither**

$\boxed{\textbf{Bounded}}$ **Unbounded**

(iii) $c_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$

Circle all which apply:

$\boxed{\textbf{Monotone Increasing}}$ **Monotone Decreasing** **Neither**

Bounded $\boxed{\textbf{Unbounded}}$

b. [6 points] For each of the following sequences or series described below, determine whether they must converge, must diverge, or whether there is not enough information. Circle your answers. No justification is required.

(i) A sequence given by $d_n = P(1-n)$ where $P(x)$ is a cumulative distribution function.

Circle one: $\boxed{\textbf{Converges}}$ **Diverges** **Not Enough Information**

(ii) An increasing sequence of positive numbers which are all smaller than 4.

Circle one: $\boxed{\textbf{Converges}}$ **Diverges** **Not Enough Information**

(iii) An infinite geometric series whose 2022nd term is 102 and whose 2023rd term is -204.

Circle one: **Converges** $\boxed{\textbf{Diverges}}$ **Not Enough Information**

7. [18 points] Determine if the following series converge absolutely, converge conditionally, or diverge. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

a. [8 points] $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$

Circle one: $\boxed{\textbf{Converges Absolutely}}$ **Converges Conditionally** **Diverges**

Solution:
Note first that $\dfrac{\sin(4n)}{4^n}$ is not always positive, so we should not try to apply the Comparison Test directly. Instead, consider the series $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin(4n)}{4^n}\right|$.

For $n \geq 1$, we have $\left|\dfrac{\sin(4n)}{4^n}\right| \leq \dfrac{1}{4^n}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n}$ converges as it is a geometric series with common ratio $\frac{1}{4}$.

Therefore, by the (Direct) Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin(4n)}{4^n}\right|$ converges. This means that the original series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$ converges absolutely.

7. (continued) Here is a reproduction of the instructions for this problem:
Determine if the following series converge absolutely, converge conditionally, or diverge. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

b. [10 points] $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ \ln(n)}$

Circle one: **Converges Absolutely** $\boxed{\textbf{Converges Conditionally}}$ **Diverges**

Solution: Note that the series is alternating. Let $a_n = \dfrac{1}{n\ \ln(n)}$. Then for all n, $0 < a_{n+1} < a_n$, and we also have $\displaystyle\lim_{n\to\infty} a_n = 0$. Therefore, by the alternating series test, $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ \ln(n)}$ converges.

Now consider the series $\displaystyle\sum_{n=3}^{\infty} \left|\frac{(-1)^n}{n\ \ln(n)}\right| = \sum_{n=3}^{\infty} \frac{1}{n\ \ln(n)}$.

Let $f(x) = \dfrac{1}{x\ \ln(x)}$. Then $f(x)$ is positive and decreasing. We have:

$$\begin{aligned}
\int_3^{\infty} \frac{1}{x\ \ln(x)}\, \mathrm{d}x &= \lim_{b\to\infty} \int_3^b \frac{1}{x\ \ln(x)}\, \mathrm{d}x \\
&= \lim_{b\to\infty} \int_3^b \frac{1}{x\ \ln(x)}\, \mathrm{d}x \\
&= \lim_{b\to\infty} \int_{\ln(3)}^{\ln(b)} \frac{1}{u}\, \mathrm{d}u \\
&= \lim_{b\to\infty} \ln(u)\Big|_{\ln(3)}^{\ln(b)} \\
&= \lim_{b\to\infty} \left(\ln(\ln(b)) - \ln(\ln(3))\right) = \infty.
\end{aligned}$$

Therefore, $\displaystyle\int_3^{\infty} \frac{1}{x\ \ln(x)}\, \mathrm{d}x$ diverges, and so by the Integral test, $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ \ln(n)}$ diverges too.

Therefore $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ \ln(n)}$ converges conditionally.

8. [12 points]

a. [7 points] Determine the **radius** of convergence for the following power series. Show all of your work. You do not need to find the interval of convergence.

$$\sum_{n=1}^{\infty}(-1)^n \frac{(2n)!}{9^n (n!)^2} x^{3n}$$

Solution: We use the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{9^n}{9^{n+1}}\cdot\frac{(2n+2)!}{(2n)!}\cdot\frac{(n!)^2}{((n+1)!)^2}\cdot\left|\frac{x^{3n+3}}{x^{3n}}\right| \\ &= \lim_{n\to\infty}\frac{1}{9}\cdot\frac{(2n+2)(2n+1)}{(n+1)^2}\cdot\left|x^3\right| \\ &= \frac{4}{9}\left|x^3\right|.\end{aligned}$$

The ratio test tells us the power series converges when this value is smaller than 1, i.e. $\frac{4}{9}\left|x^3\right| < 1$. Rearranging, we see that this implies $|x| < \left(\frac{9}{4}\right)^{1/3}$, which tells us that the radius of convergence is $\left(\frac{9}{4}\right)^{1/3}$.

Answer: $\left(\frac{9}{4}\right)^{1/3}$

b. [5 points] No justification is needed for the remainder of this problem. Suppose that the following is true about the sequence C_n which is defined for $n \geq 0$:

- C_n is monotone decreasing and converges to 0.
- $\displaystyle\sum_{n=0}^{\infty} C_n$ diverges.
- The power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n C_n}{6^n}(x-5)^n$ has radius of convergence 6.

What is the center of the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n C_n}{6^n}(x-5)^n$?

Answer: 5

What are the endpoints of the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n C_n}{6^n}(x-5)^n$?

Answer: Left endpoint at $c =$ -1

Right endpoint at $d =$ 11

Let c and d be the left and right endpoints of the interval of convergence you found above. Which of the following could be the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n C_n}{6^n}(x-5)^n$? Circle **all** correct answers.

(c, d) $\qquad$ $\boxed{(c, d]}$ $\qquad$ $[c, d)$ $\qquad$ $[c, d]$

9. [12 points] For the following questions, determine if the statement is ALWAYS true, SOMETIMES true, or NEVER true, and circle the corresponding answer. Justification is not required.

a. [2 points] If the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ diverges, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ also diverges.

Circle one: **ALWAYS** $\boxed{\textbf{SOMETIMES}}$ **NEVER**

b. [2 points] If b_n is a sequence of positive numbers which satisfy $\displaystyle\lim_{n\to\infty}\frac{1}{n^3 b_n} = 12$, then $\displaystyle\sum_{n=1}^{\infty} b_n$ converges.

Circle one: $\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

c. [2 points] If $f(x)$ is a continuous function so that $\displaystyle\int_0^{\infty} f(x)\,\mathrm{d}x$ converges, then $\displaystyle\int_{10}^{\infty}\left(f(x)+\frac{1}{x^5}\right)\mathrm{d}x$ converges too.

Circle one: $\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

d. [2 points] If $\displaystyle\sum_{n=0}^{\infty} d_n = \frac{1}{1-0.3}$, then $d_n = (0.3)^n$ for all $n \geq 0$.

Circle one: **ALWAYS** $\boxed{\textbf{SOMETIMES}}$ **NEVER**

e. [2 points] The function given by

$$g(x) = \begin{cases} x^3, & -1 \leq x \leq \sqrt[4]{5}, \\ 0, & \text{otherwise}, \end{cases}$$

is a probability density function.

Circle one: **ALWAYS** **SOMETIMES** $\boxed{\textbf{NEVER}}$

f. [2 points] If s_n is a decreasing sequence of positive numbers which converges, then $\displaystyle\sum_{n=1}^{\infty} s_n$ converges too.

Circle one: **ALWAYS** $\boxed{\textbf{SOMETIMES}}$ **NEVER**