Math 116 — Final Exam — December 8, 2023

Write your 8-digit UMID number very clearly in the box to the right.

Your Initials Only: ______ Instructor Name: ______________________ Section #: ____

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 13 pages including this cover.
3. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	8	
3	12	
4	6	
5	7	
6	10	

Problem	Points	Score
7	16	
8	9	
9	8	
10	12	
Total	100	

1. [12 points] Compute the exact value of each of the following, if possible. Your answers should not involve integration signs, ellipses or sigma notation. For any values which do not exist, write **DNE**. You do not need to show work.

a. [2 points] The integral $\int_{-10}^{10} (f(x) + 1)\ \mathrm{d}x$, where $f(x)$ is an odd function.

Answer: 20

b. [2 points] The integral $\int_{-3}^{4} \frac{1}{x^4}\ \mathrm{d}x$.

Answer: **DNE**

c. [2 points] The sum $\sum_{n=0}^{2023} 7(5)^n$.

Answer: $\frac{7(1-5^{2024})}{1-5} = \frac{7}{4}(5^{2024}-1)$

d. [2 points] The **radius of convergence** for the Taylor series centered around $x = 0$ for the function $g(x) = (1 + 3x^2)^{1/5}$.

Answer: $\frac{1}{\sqrt{3}}$

e. [2 points] The infinite sum $(0.5)^2 - \frac{(0.5)^4}{2} + \frac{(0.5)^6}{3} - \cdots + \frac{(-1)^{n+1}(0.5)^{2n}}{n} + \cdots$.

Answer: $\ln\left(\frac{5}{4}\right)$

f. [2 points] The value of $h''(2)$ where the fourth-degree Taylor polynomial for $h(x)$ about $x = 2$ is given by $P_4(x) = 2 + 9(x-2) - 81(x-2)^4$.

Answer: **0**

2. [8 points] Consider the function $G(x) = x^3\cos(2x)$.

a. [4 points] Give the first four nonzero terms of the Taylor series of $G(x)$ centered about $x = 0$.

Solution: Using the known Taylor series of $\cos(x)$ centered at $x = 0$, we have

$$x^3\cos(2x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2x)^{2n}x^3 = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}x^{2n+3}.$$

Thus, the first four nonzero terms of the Taylor series of $G(x)$ about $x = 3$ are

$$x^3 - \frac{2^2}{2!}x^5 + \frac{2^4}{4!}x^7 - \frac{2^6}{6!}x^9.$$

Answer: $x^3 - \frac{2^2}{2!}x^5 + \frac{2^4}{4!}x^7 - \frac{2^6}{6!}x^9$

b. [4 points] Find $G^{(2023)}(0)$. You do not need to simplify.

Solution: The 2023rd power of x will appear in the Taylor series expansion of $G(x)$ with a nonzero coefficient. We can see this by noting that $2n + 3 = 2023$ if $n = 1010$. Therefore,

$$\frac{G^{(2023)}(0)}{2023!} = \frac{(-1)^{1010}\,2^{2\cdot 1010}}{(2\cdot 1010)!},$$

so $G^{(2023)}(0) = (2023)! \cdot \dfrac{(-1)^{1010}\,2^{2020}}{2020!}$.

Answer: $(2023)! \cdot \dfrac{(-1)^{1010}\,2^{2020}}{2020!}$

3. [12 points] Antonia the ant is entering her first bug race. The track runs from the start line at the south end, represented by $y = 0$, to the finish line at the north end, represented by $y = 4$. All distances are given in feet.
Antonia's position t seconds after the race begins is given in parametric equations by:

$$x = \sin\left(\frac{\pi t}{2}\right), \qquad y = 1.5^t - 1,$$

a. [2 points] What is Antonia's position 2 seconds into the race?

Solution: To find this, we plug $t = 2$ into the equations given for the x and y coordinates of Antonia. We find $x = \sin(\pi) = 0$ and $y = 1.5^2 - 1 = 1.25$.

$x =$ ___0___ $y =$ ___1.25___

b. [3 points] At what time does Antonia reach the finish line?

Solution: We set $y = 4$ to to get $1.5^t - 1 = 4$. Therefore, $1.5^t = 5$, and so $t = \dfrac{\ln(5)}{\ln(1.5)}$.

The time is $t =$ ___$\dfrac{\ln(5)}{\ln(1.5)}$___

c. [3 points] What is the first time during the race that Antonia is travelling directly north?

Solution: Note that y is always increasing, so we only need to find the first time that $\dfrac{dx}{dt} = 0$.
We have $\dfrac{dx}{dt} = \dfrac{\pi}{2}\cos\left(\dfrac{\pi t}{2}\right)$, which is equal to 0 when $t = 1$.

The time is $t =$ ___1___

d. [4 points] Write an expression involving one or more integrals that gives the total distance, in feet, that Antonia traveled during the race. Do not evaluate your integral(s).

Solution: We use the arclength formula, with $\dfrac{dx}{dt} = \dfrac{\pi}{2}\cos\left(\dfrac{\pi t}{2}\right)$, $\dfrac{dy}{dt} = \ln(1.5)1.5^t$, and using the time we found in part b.

The distance is ___$\displaystyle\int_0^{\ln(5)/\ln(1.5)} \sqrt{\left(\frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)\right)^2 + (\ln(1.5)1.5^t)^2}\, dt$___

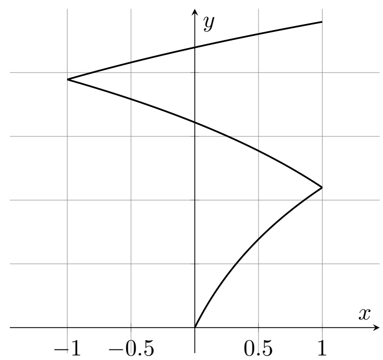
4. [6 points]

a. [3 points] Bertie the beetle enters a different bug race. This race lasts for a thrilling 10 seconds. His position t seconds into his race is given by $(x(t), y(t))$ where

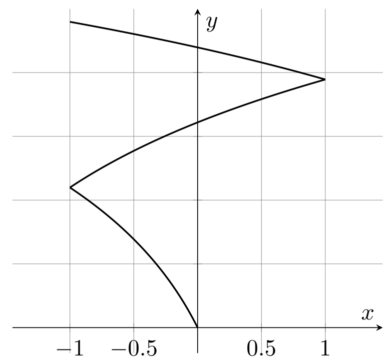
$$x(t) = \begin{cases} \frac{1}{2}t, & 0 \le t < 2, \\ \frac{1}{2}(4-t), & 2 \le t < 6, \\ \frac{1}{2}(t-8), & 6 \le t < 10, \end{cases}$$

and where $y(t) = \ln(t+1)$. Which of the following graphs best represents the path he follows? Circle the **one** best answer.

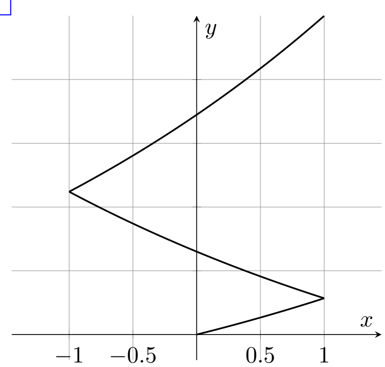
(**A**)



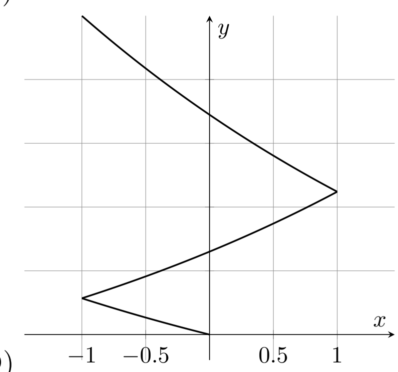
(**B**)



(**C**)



(**D**)



b. [3 points] Carlos the centipede is training for a bug race by running in circular laps. On his first lap, Carlos' position t seconds after he began his lap is given by the parametric equations $x = f(t)$ and $y = g(t)$. For the second lap, his position t seconds after the lap begins is given by the parametric equations $x = f(2t)$ and $y = g(2t)$.
How is Carlos' path in the second lap different from the first? Circle the **one** best answer from the options below.

(I) Carlos follows a path which has the same shape as the one for the first lap, but which has half the diameter.

(II) Carlos follows a path which has the same shape as the one for the first lap, but which has twice the diameter.

(III) Carlos follows the same path as before but travels at half the speed.

(IV) Carlos follows the same path as before but travels at twice the speed.

5. [7 points] A local beet company, Dope Beets Inc., is developing a new beet with an adjustable growth rate for its many different customers. The growth rate of their new beet, measured in pounds per day, t days after a beet is planted, is given by

$$r(t) = \frac{5t^2}{t^k + t + 1},$$

for some adjustable value $k > 1$.

a. [4 points] Suppose a new beet initially weighs 2 pounds. Write an expression involving an integral for the weight, in pounds, of the beet t days after it is planted.

Answer: $2 + \int_0^t \frac{5x^2}{x^k + x + 1}\, dx$

b. [3 points] Dope Beets Inc. wants to adjust the value of k such that a planted beet will never have infinite weight, even if the beet is allowed to grow forever. Which values of k would keep the weight finite? Give your answer as a value, list of values, or interval, as appropriate. No justification is required.

Solution: (Not required). Consider the integral

$$\int_1^\infty \frac{1}{x^{k-2}}\, dx.$$

Let $p = k - 2$; this integral converges by the p-test when $p > 1$, or when $k > 3$. By (properly) using the direct or limit comparison tests, this shows the $\int_0^\infty \frac{5x^2}{x^k + x + 1}\, dx$ (the total change in a beet's weight over all time) converges to a finite value.

Answer: $k > 3$

6. [10 points] Consider an infinitely differentiable function $f(x)$. The following table gives some values of $f(x)$ and its derivatives at $x = 1$:

$f(1)$	$f'(1)$	$f''(1)$	$f'''(1)$
$\pi/4$	$1/2$	$-1/4$	2

a. [4 points] Write down $P_3(x)$, the third-degree Taylor polynomial of $f(x)$ about $x = 1$. You do not need to simplify.

$P_3(x) =$ $\dfrac{\pi}{4} + \dfrac{1}{2}(x-1) - \dfrac{1}{4\cdot 2!}(x-1)^2 + \dfrac{2}{3!}(x-1)^3$

b. [3 points] Recall that $f(x) \approx P_3(x)$ near $x = 1$. Use this and the fact that $f(1.5) = \pi/3$ to write an approximation for π. You do not need to simplify your answer. Your answer should not contain the symbol π.

Solution: Using $f(x) \approx P_3(x)$ near $x = 1$, we have

$$\frac{\pi}{3} = f(1.5) \approx P_3(1.5) = \frac{\pi}{4} + \frac{1}{2}(0.5) - \frac{1}{4\cdot 2!}(0.5)^2 + \frac{2}{3!}(0.5)^3.$$

So $\frac{\pi}{3} - \frac{\pi}{4} \approx \frac{1}{2}(0.5) - \frac{1}{4\cdot 2!}(0.5)^2 + \frac{2}{3!}(0.5)^3$. By getting a common denominator of 12 on the left-hand side of the approximation, we get

$$\pi \approx 12 \cdot \left(\frac{1}{2}(0.5) - \frac{1}{4\cdot 2!}(0.5)^2 + \frac{2}{3!}(0.5)^3\right).$$

$\pi \approx$ $12 \cdot \left(\frac{1}{2}(0.5) - \frac{1}{4\cdot 2!}(0.5)^2 + \frac{2}{3!}(0.5)^3\right)$

c. [3 points]

Use the Taylor polynomial from part **a**. to approximate the definite integral

$$\int_1^{1.1} f(x)\,\mathrm{d}x.$$

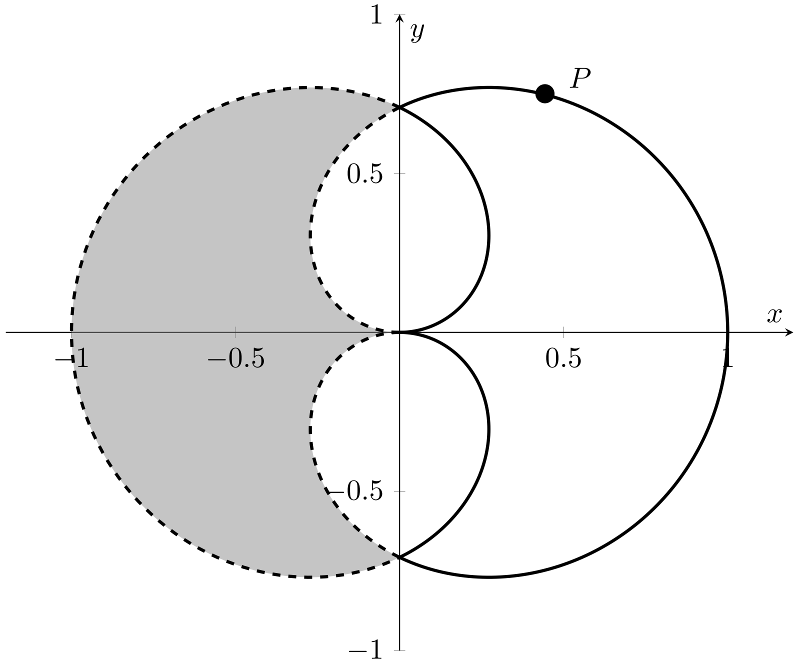
You do not need to simplify your answer.

Solution:

$$\int_1^{1.1} f(x)\,\mathrm{d}x \approx \int_1^{1.1} P_3(x)\,\mathrm{d}x = \frac{\pi}{4}(0.1) + \frac{1}{2\cdot 2}(0.1)^2 - \frac{1}{3\cdot 4\cdot 2!}(0.1)^3 + \frac{2}{4\cdot 3!}(0.1)^4.$$

Answer: $\frac{\pi}{4}(0.1) + \frac{1}{2\cdot 2}(0.1)^2 - \frac{1}{3\cdot 4\cdot 2!}(0.1)^3 + \frac{2}{4\cdot 3!}(0.1)^4$

7. [16 points] A particle moves along a path given by the polar curve $r = \cos(\theta/2)$, $0 \le \theta \le 4\pi$. The polar curve is graphed below. A portion of the polar curve is dashed.



a. [4 points] The distance from the origin to the point labeled P is $\sqrt{3}/2$. Find the Cartesian coordinates corresponding to the point labeled P.

Solution: Consider the polar coordinates (r, θ) of the point P. By the hypothesis of the problem, $r = \sqrt{3}/2$. To determine θ, we want to solve $\cos(\theta/2) = \sqrt{3}/2$. We find that

$$\theta/2 = \pi/6 \Rightarrow \theta = \pi/3.$$

Plugging the polar coordinates into $x = r\cos(\theta), y = r\sin(\theta)$ gets us

$$x = \frac{\sqrt{3}}{2}\cos(\pi/3) = \frac{\sqrt{3}}{4}, \quad y = \frac{\sqrt{3}}{2}\sin(\pi/3) = \frac{3}{4}$$

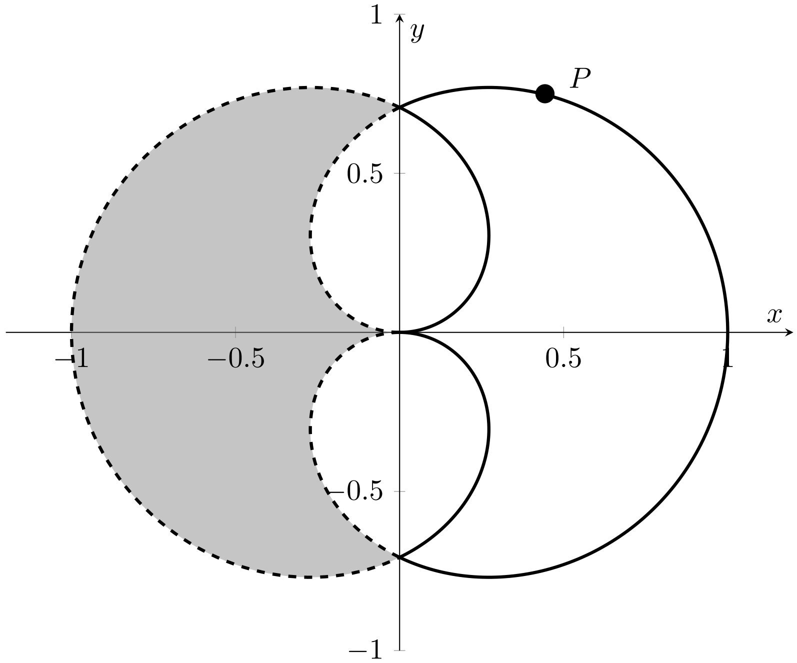
$(x, y) =$ $(\sqrt{3}/4, 3/4)$

b. [4 points] For what values of θ in $[0, 4\pi]$ does the particle pass through the origin?

Solution: We solve $\cos(\theta/2) = 0$ and see that $\theta/2 = m\pi/2$, where m is an odd integer. The only θ in the interval $[0, 4\pi]$ that satisfy this equation are $\theta = \pi, 3\pi$.

$\theta =$ $\pi, 3\pi$

7. (continued) The graph of the polar curve $r = \cos(\theta/2)$, with $0 \le \theta \le 4\pi$, from the previous page is reproduced below:



c. [4 points] Determine the interval(s) within $[0, 4\pi]$ for which θ traces out the **dashed** portion of the graph.

Answer: $[\pi/2, \pi] \cup [3\pi/2, 5\pi/2] \cup [3\pi, 7\pi/2]$

d. [4 points] Write an expression involving one or more integrals for the shaded area enclosed by the dashed portion of the particle's path. Do not evaluate your integral(s).

Answer: $\displaystyle\int_{3\pi/2}^{5\pi/2} \frac{1}{2}\cos^2(\theta/2)\,\mathrm{d}\theta - 2\int_{\pi/2}^{\pi} \frac{1}{2}\cos^2(\theta/2)\,\mathrm{d}\theta$

8. [9 points] Gabriella is developing a new kind of vuvuzela. In order to come up with a new method, she first considers the old way she made her instruments.

a. [4 points] Gabriella initially made her vuvuzelas by considering a positive function $f(x)$, and forming a region $\mathcal{R}$ between $y = f(x)$ and the x-axis on the interval $[2, \infty)$. She rotated $\mathcal{R}$ about the x-axis to form the shape of the vuvuzela. Write an integral which gives the volume of the vuvuzela. Your answer will involve the function $f(x)$.

Answer: $\int_2^\infty \pi(f(x))^2\,\mathrm{d}x$

b. [5 points] For her new batch of vuvuzelas, Gabriella considers an entirely different shape. The volume of the new design of vuvuzela is given by

$$\int_2^\infty \frac{x}{(x^2+5)^2}\,\mathrm{d}x.$$

Compute the value of this integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

Solution: We begin by writing the improper integral as a corresponding limit. Using a substitution of $u = x^2 + 5$, we have

$$\begin{aligned}
\int_2^\infty \frac{x}{(x^2+5)^2}\,\mathrm{d}x &= \lim_{b\to\infty} \int_2^b \frac{x}{(x^2+5)^2}\,\mathrm{d}x \\
&= \lim_{b\to\infty} \int_9^{b^2+5} \frac{(1/2)}{u^2}\,\mathrm{d}u \\
&= \lim_{b\to\infty} -\frac{1}{2(b^2+5)} + \frac{1}{2(9)} \\
&= -0 + \frac{1}{18}.
\end{aligned}$$

Therefore the integral converges to $\frac{1}{18}$.

Circle one: **Diverges** **Converges to** $\frac{1}{18}$

9. [8 points] A power series centered at $x = 3$ given by

$$\sum_{n=0}^{\infty} \frac{n^2+1}{2^n(n+2)}(x-3)^n.$$

The radius of convergence of this power series is 2 (do NOT show this). Find the **interval** of convergence of this power series. Show all your work, including full justification for series behavior.

Solution: Since we know the radius of convergence, we just need to test the behavior at the endpoints, which are $3 \pm 2 = 1, 5$. At $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{n^2+1}{2^n(n+2)}(1-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(n^2+1)}{n+2}.$$

To determine the behavior of this, we use the nth term test for divergence.

$$\lim_{n\to\infty} \frac{(-1)^n(n^2+1)}{n+2} = \lim_{n\to\infty} (-1)^n n.$$

which does not exist, and so the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n^2+1)}{n+2}$ diverges by the nth term test for divergence. So $x = 1$ is not included in the interval of convergence.

Similarly, at $x = 5$, the series is

$$\sum_{n=0}^{\infty} \frac{n^2+1}{2^n(n+2)}(5-3)^n = \sum_{n=0}^{\infty} \frac{n^2+1}{n+2}.$$

Since $\displaystyle\lim_{n\to\infty} \frac{n^2+1}{n+2} = \infty$, this series also diverges by the nth term test for divergence. Therefore, the interval of convergence is $(1, 5)$.

Interval of convergence: $(1, 5)$

10. [12 points] For the following questions, determine if the statement is ALWAYS true, SOMETIMES true, or NEVER true, and circle the corresponding answer. Justification is not required.

a. [2 points] Suppose $H(x)$ is a continuous function such that $H'(x) > 0$ and $H(x) \geq 0$ for all x. Then $H(x)$ is a cumulative distribution function (cdf).

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

b. [2 points] If a_n is a sequence of positive numbers, and the sequence $S_n = a_1 + \cdots + a_n$ converges to S, then a_n converges to S.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

c. [2 points] The average value of a continuous function $f(x)$ on the interval $[0, 1]$ is given by $\displaystyle\int_0^1 xf(x)\,dx$.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

d. [2 points] $\displaystyle\int_2^3 \frac{1}{x\ln(x)}\,dx = \int_2^3 \frac{1}{u}\,du.$

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

e. [2 points] If n is a fixed number which is bigger than 100, and MID(n) and LEFT(n) both estimate $\displaystyle\int_0^{\pi/2} \cos(x)\,dx$, then

$$\int_0^{\pi/2} \cos(x)\,dx \leq \text{MID}(n) \leq \text{LEFT}(n).$$

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

f. [2 points] If $r = f(\theta)$ is a polar curve, then the arclength of the part of the curve in the first quadrant is given by $\displaystyle\int_0^{\pi/2} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\,d\theta$.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin\theta$	$\cos\theta$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$